Contributions to the Koopman Theory of Dynamical Systems

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Summary

German

In der vorliegenden Arbeit wird die Theorie der sogenannten Koopmanlinearisierung dynamischer Systeme erweitert, die in der Monographie „Operator Theoretic Aspects of Ergodic Theory“[1] verfolgt wird. Anstatt die Dynamik auf dem Zustandsraum selbst zu betrachten, untersucht man dabei den induzierten linearen beschränkten Operator auf geeigneten Banachräumen von Observablen. Dieser Perspektivwechsel ermöglicht den Einsatz von funktionalanalytischen und operatortheoretischen Methoden, um Fragestellungen und Probleme der topologischen Dynamik und der Ergodentheorie anzugehen.

Im ersten Artikel werden verschiedene Kompaktifizierungen von Operatorhalbgruppen betrachtet. Dabei werden die in der topologischen Dynamik auftretetenden kompakten Halbgruppen (Ellis-, Köhler- und Jacobshalbgruppen) in eine systematische Theorie eingebettet. Der Zusammenhang zwischen konvexen kompakten rechtstopologischen Halbgruppen und Mittelergodizität wird in einem sehr allgemeinen Rahmen untersucht und dann konkret auf die topologische Dynamik angewandt. Ein besonderer Fokus liegt dabei auf der Klasse der zahmen dynamischen Systeme.

Der zweite Teil der Arbeit befasst sich mit dem Primitivspektrum einer Markovhalbgruppe $\mathcal{S}$ auf dem Banachverband $\mathrm{C}(K)$, wobei K ein kompakter Raum ist. Aufbauend auf Artikeln von H. H. Schaefer werden bestimmte invariante Ideale betrachtet und, in Analogie zum Primitivspektrum von C*-Algebren bzw. dem Primspektrum kommutativer Ringe in der algebraischen Geometrie, topologisiert. Der gewonnene topologische Raum beschreibt die Struktur des Fixraumes von $\mathcal{S}$ und erlaubt eine neue Charakterisierung der Mittelergodizität von $\mathcal{S}$. Weiterhin werden sogenannte minimale Anziehungszentren in K mit der Idealstruktur von $\mathrm{C}(K)$ in Verbindung gesetzt.

Im dritten Manuskript wird eine systematische Koopmanlinearisierung für Dynamiken auf topologischen und messbaren Banachbündeln entwickelt. Die zentralen Resultate sind algebraische und verbandstheoretische Charakterisierungen der induzierten gewichteten Koopmanoperatoren auf den zugehörigen Räumen von Schnitten. Diese können als Ausgangspunkt für eine operatortheoretische Untersuchung von Kozykeln über Flüssen dienen.

Der letzte Teil der Arbeit beschäftigt sich mit strukturierten Erweiterungen dynamischer Systeme und deren operatortheoretischer Charakterisierung. Dabei werden Erweiterungen topologischer

[1] [EFHN15]

dynamischer Systeme sogenannte einhüllende Semigruppoide zugeordnet. Im Spezialfall pseudoisometrischer Erweiterungen erhält man dadurch kompakte Gruppoide, auf die unter geeigneten Voraussetzungen eine Darstellungstheorie anwendbar ist. Dies liefert für eine große Klasse von Systemen die gewünschte Beschreibung strukturierter Erweiterungen durch Eigenschaften des Koopmanoperators.

English

In this thesis we extend the theory of the so-called Koopman linearization of dynamical systems as systematically pursued in the monograph "Operator Theoretic Aspects of Ergodic Theory"[2]. Instead of considering the dynamics on the state space itself, one investigates the induced linear bounded operator on suitable Banach spaces of observables. This change of perspective opens the door to use functional analytic and operator theoretic methods to address questions and problems of topological dynamics and ergodic theory.

In the first article various compactifications of operator semigroups are considered. Compact semigroups (Ellis, Köhler and Jacobs semigroups) appearing in topological dynamics are integrated into a systematic theory. The connection between convex compact right topological semigroups and mean ergodicity is examined in a very general setting and applied to topological dynamics. A special focus is laid on the class of tame dynamical systems.

The second part of the thesis is concerned with the primitive spectrum of Markov semigroups $\mathcal{S}$ on the Banach lattice $\mathrm{C}(K)$ where K is a compact space. Based on articles of H. H. Schaefer certain invariant ideals are considered and topologised in analogy to the primitive spectrum of C*-algebras or the prime spectrum of commutative rings in algebraic geometry. The resulting topological space describes the structure of the fixed space of $\mathcal{S}$ and yields a new characterization of mean ergodicity of $\mathcal{S}$. In addition, a connection between so-called minimal centers of attraction in K and the ideal structure of $\mathrm{C}(K)$ is revealed.

In the third manuscript a systematic Koopman linearization for dynamics on topological and measurable Banach bundles is developed. The main results are algebraic and lattice theoretic characterizations of the induced weighted Koopman operators on the asscodiated spaces of sections. These can be the starting point for an operator theoretic investigation of cocycles over flows.

The last part of the thesis deals with structured extensions of dynamical systems and their operator theoretic characterization. In the process enveloping semigroupoids are associated to extensions of topological dynamical systems. In the special case of pseudoisometric extensions one obtains compact groupoids for which—under suitable assumptions—a representation theory is applicable. For a large class of systems this yields the desired description of structured extensions via properties of the Koopman operator.

[2] [EFHN15]

List of Publications

Accepted Manuscripts

Compact operator semigroups applied to dynamical systems by Henrik Kreidler, published in *Semigroup Forum*, 2018, volume 97, pages 523–547 and cited as [Kre18] in this thesis. The peer reviewed version accepted for publication is contained in Appendix 1.1. The final authenticated version is available online at: https://doi.org/10.1007/s00233-018-9958-x

The primitive spectrum of a semigroup of Markov operators by Henrik Kreidler, published online in *Positivity*, 2019 and cited as [Kre19] in this thesis. The peer reviewed version accepted for publication is contained in Appendix 1.2. The final authenticated version is available online at: https://doi.org/10.1007/s11117-019-00678-0

Submitted Manuscripts

Gelfand-type theorems for dynamical Banach modules by Henrik Kreidler and Sita Siewert. Submitted to *Mathematische Zeitschrift*, 2019. Cited as [KS19] in this thesis. The submitted version of the article is contained in Appendix 2.1.

Additional Manuscripts

Uniform enveloping semigroupoids for extensions of topological dynamical systems by Nikolai Edeko and Henrik Kreidler. In preparation, 2019. Cited as [EK19] in this thesis. A preliminary version of the article is contained in Appendix 3.1.

Personal Contribution

Accepted Manuscripts

The articles *Compact operator semigroups applied to dynamical systems* ([Kre18]) and *The primitive spectrum of a semigroup of Markov operators* ([Kre19]) have been written by myself entirely. Earlier versions of some results of [Kre18] are contained in my Master thesis [Kre16] (in particular, parts of Sections 3 and 5 of the article) but have been generalized and their proofs have been reworked. Moreover, Roland Derndinger, Nikolai Edeko, Rainer Nagel, Alexander Romanov and the anonymous referee reviewing the paper provided helpful advice and ideas. For the article [Kre19] Roland Derndinger, Nikolai Edeko, Ulrich Groh, Rainer Nagel and the anonymous referee reviewing the paper contributed their ideas and suggestions.

For both articles an estimated 95% of the scientific ideas and 100% of the writing are due to myself.

Submitted Manuscripts

The article *Gelfand-type theorems for dynamical Banach modules* ([KS19]) is joint work with Sita Siewert. All results of the paper were formulated, discussed and proved in cooperation. Discussions with Nikolai Edeko, Daniel Hättig, Viktoria Kühner, Philipp Kunde, Marco Peruzzetto, Walther Paravicini and Marco Peruzzetto inspired us when writing the article.

I contributed an estimated 47, 5% of the scientific ideas and 50% of the writing to this article.

Additional Manuscripts

The manuscript *Uniform enveloping semigroupoids for extensions of topological dynamical systems* ([EK19]) is joint work with Nikolai Edeko. All results of the paper were formulated, discussed and proved in cooperation. Discussions with Markus Haase, Rainer Nagel and Jean Renault have been very helpful for this article.

I contributed an estimated 47, 5% of the scientific ideas and 50% of the writing to this article.

1 Introduction: What is Koopmanism?

The beginnings of ergodic theory and the mathematical theory of dynamical systems go all the way back to the end of the 18th century and Ludwig Boltzmann's contributions to thermodynamics[1]. He considered gases and fluids by examining the behavior of their particles. The state of such physical dynamical systems can, at a given time, be modelled by the positions and momenta of all involved atoms or molecules. It can thus be considered as a point in $\mathbb{R}^{6N}$ for some (large) natural number N and the collection of all possible states can then be seen as a subset Ω of the space $\mathbb{R}^{6N}$. This is what usually is called the *state space* of the given system. Now, after one time step, the particles have moved leading to a new state. In the mathematical model this process corresponds to a mapping $\varphi\colon \Omega \to \Omega$ assigning to a given state a new one.

Examining dynamical systems

Using such a model as a starting point, we ask questions about its properties: What is its long–time behavior? Does the system converge to some kind of equilibrium? Can every possible state be (at least approximately) reached if we wait long enough? Or is there a set of states the system gravitates to, i.e., some sort of attractor? Can we single out "structured parts"?

These and other questions arise naturally and are of importance for engineers working, e.g., with fluid dynamics or aerodynamics. Translated into mathematical language they can be examined with all the tools of modern mathematics. In fact, this has been done in the past and lead—by imposing additional structure on $(\Omega;\varphi)$—to theories for topological, differentiable and measure-preserving dynamical systems (see, e.g., [GH55], [Bro70], [Par81] [Man87], [Gla03] and [BP13]).

Observables and the idea of Koopmanism

Let us, however, return to the physical viewpoint once more. When considering the particle movement of a gas or fluid, it is impossible to know the exact state of the whole system at a given time. Instead one has to be content with making measurements of, e.g., pressure or temperature, and observe how these "observables" change. While this approach is a necessity in physics, the change of perspective also has merits for mathematics and leads to the approach now called *Koopmanism*.

[1] See [Mat88], [Bad06] or the introduction of [EFHN15] for short historical accounts of the origins of ergodic theory.

In our mathematical model, an *observable* is a complex-valued function $f : \Omega \to \mathbb{C}$, and considering its change after a time step amounts to looking at a new observable $T_\varphi f := f \circ \varphi$. If we regard the space $\mathrm{F}(\Omega) := \mathbb{C}^\Omega$ of all observables, this gives rise to a mapping $T_\varphi : \mathrm{F}(\Omega) \to \mathrm{F}(\Omega)$. The set $\mathrm{F}(\Omega)$ can be turned into a vector space and even an algebra over $\mathbb{C}$ with the natural operations and T_φ is then an algebra homomorphism. Likewise, $\mathrm{F}(\Omega)$ is—with the canonical order—a complex vector lattice and T_φ then is a lattice homomorphism.

Even though these observations are simple, we have gained much algebraic and order theoretic structure in the process. Most importantly, we have constructed a *linear* map T_φ out of $(\Omega; \varphi)$. This is the *Koopman operator*[2]. Note that in contrast to local linearizations of dynamics around fixed points this is a global linearization of φ.

Variants of Koopmanism

Inducing linear operators on function spaces today is a common mathematical method and, depending on the context, the operator T_φ is called the *induced operator*, *composition operator* or simply the *pullback* of φ. It appears in differential geometry (differentiable mappings induce pullbacks on spaces of smooth functions), algebraic geometry (morphisms of algebraic varieties induce pullbacks on spaces of regular functions) and complex dynamics (holomorphic mappings induce composition operators on Banach and Fréchet spaces of holomorphic functions, see, e.g., [CM95]). In the theory of C*-dynamics the concept of *Quantum Koopmanism* has been proposed (see pages 161–165 of [AJP06]). We focus on the Koopman linearizations in the context of topological dynamics and ergodic theory.

Consider a *topological dynamical system* $(K; \varphi)$, i.e., a compact space K and a continuous mapping $\varphi : K \to K$. Equipped with the supremum norm, the subalgebra and sublattice $\mathrm{C}(K) \subseteq \mathrm{F}(K)$ of continuous complex-valued functions is then a Banach space. It is clear that T_φ restricts to a contraction on $\mathrm{C}(K)$ and we therefore obtain a Banach space $E = \mathrm{C}(K)$ and a bounded operator $T = T_\varphi \in \mathscr{L}(\mathrm{C}(K))$ on that space.

In ergodic theory the objects of interest are *measure-preserving systems*, i.e., one considers a probability space $X = (\Omega_X, \Sigma_X, \mu_X)$ consisting of a set Ω_X, a σ-algebra Σ_X of subsets of Ω_X and a probability measure $\mu_X : \Sigma_X \to [0, 1]$ as well as a measurable mapping $\varphi : \Omega_X \to \Omega_X$ such that $\mu_X(\varphi^{-1}(A)) = \mu_X(A)$ for every $A \in \Sigma_X$. The Koopman operator T_φ then leaves the space of measurable functions invariant and induces an operator on the spaces $\mathrm{L}^p(X)$ for every $p \in [1, \infty]$ also denoted by T_φ. Again we obtain what we call a *functional analytic dynamical system*, i.e., a pair $(E; T)$ of a Banach space E and a bounded operator $T \in \mathscr{L}(E)$ on this space.

In both cases we therefore have the full toolbox of functional analysis and operator theory at our disposal in order to investigate φ through the induced Koopman operator T_φ.

[2]Named after Bernard Koopman, a coauthor of John von Neumann, who introduced the operator on L^2-spaces in his article [Koo31].

Recovering the system

It is of course a crucial question to what extent the functional analytic dynamical system reflects the original system. How do properties of φ translate into properties of T_φ and vice versa? And—given the functional analytic dynamical system—can we reconstruct, up to isomorphy, the original system?

In the topological situation it turns out that both—the algebraic as well as the lattice theoretic structure of $\mathrm{C}(K)$ and the operator T_φ—encode the original system. Recall that $\mathrm{C}(K)$ and T_φ are the prototypes for commutative unital C*-algebras and unital *-homomorphism on these algebras, respectively. In fact, if $(A; T)$ is a pair concisting of a commutative unital C*-algebra A and a unital *-homomorphism $T \in \mathscr{L}(A)$, then—by Gelfand's representation theorem—there are a compact space K and a continuous mapping $\varphi\colon K \to K$ such that the pair $(A; T)$ is isomorphic to $(\mathrm{C}(K); T_\varphi)$. Moreover, this system $(K; \varphi)$ is unique up to isomorphy. Note also, that the construction of $(K; \varphi)$ can be done in a canonical way by choosing K to be the Gelfand space of A contained in the dual A' of A and φ as the restriction of the adjoint T' to K. As a result, the categories of topological dynamical systems $(K; \varphi)$ and such "commutative unital C^*–dynamical systems" are anti-equivalent.

Alternatively, we can consider the pair $(\mathrm{C}(K); T_\varphi)$ in terms of Banach lattice theory. The space $\mathrm{C}(K)$ is the prototype of a so called AM-space with order unit (see Section II.7 of [Sch74] for this concept) and $T_\varphi \in \mathscr{L}(\mathrm{C}(K))$ is a Markov lattice homomorphism, i.e., a lattice homomorphism preserving the unit $\mathbb{1}$. By Kakutani's representation theorem, the assignment $(K; \varphi) \to (\mathrm{C}(K); T_\varphi)$ then also defines an anti-equivalence between the category of topological dynamical systems and "unital AM–dynamical systems".

In the measure-preserving case the situation is similar. The Koopman operator T_φ is a bi-Markov lattice homomorphism on the AL-space $\mathrm{L}^1(X)$ with quasi-interior point $\mathbb{1}$ (see Section II.8 of [Sch74]). Again these are the prototypes for such spaces and operators.[3]

We see that in both settings it is—from a category theoretic perspective—justified to look at the functional analytic dynamical systems instead of the original systems.[4]

The benefits of Koopmanism

In view of the previous results, one could argue that there is no advantage in considering the Koopman linearization since examining the functional analytic dynamical system is just as hard as looking at the original system. However, "translating" a problem of one mathematical world into a different one has proven to be very fruitful at various occasions. Concepts which are quite involved and seem complicated on one side, are easy in the other. Some ideas and constructions

[3] The situation is slightly more intricate here than in the topological situation and one has to make some separability assumptions on the probability spaces and AL-spaces. Also, constructing a (standard) probability space out of some separable AL-space with a quasi-interior point is not canonical and involves choices.

[4] As a side note: Similar equivalence results hold in algebraic geometry, see, e.g., Proposition 1.33 and Theorem 2.35 in [GW10]. A Koopman approach to differential geometry is discussed in Chapters 4 and 7 of [Nes03].

may be more intuitive in one category. And finally, new questions might arise naturally in one language, but are difficult to grasp in another. In the following we discuss advantages and applications of Koopmanism using examples from topological dynamics and ergodic theory.

Ergodic theorems. In a modern and mathematical way one can formulate the famous ergodic hypothesis of Boltzman conjecturing that "time mean equals space mean" in terms of measure-preserving systems in the following form.

Ergodic Hypothesis. *If* $(X;\varphi)$ *is an* ergodic[5] *measure-preserving system, then for every measurable set* $A \in \Sigma_X$

$$\lim_{N\to\infty} \frac{|\{n \in \{0,\dots,N-1\} \mid \varphi^n(x) \in A\}|}{N} = \mu(A)$$

for almost every $x \in \Omega_X$.

In other words: Given a measurable set, for almost every point, the average time the orbit of this point is contained in the set asymptotically is the volume of that set. While the meaning of the hypothesis is clear, only the translation of the conjecture in terms of observables and the Koopman operator made an operator theoretic investigation of the problem possible and led to the famous individual ergodic theorem of George David Birkhoff (see [Bir31]).

Theorem (Birkhoff's individual ergodic theorem)**.** *If* $(X;\varphi)$ *is a measure-preserving system and* $f \in \mathrm{L}^1(X)$*, then the limit*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} T_\varphi^n f(x)$$

exists for almost every $x \in \Omega_X$.

In a more general operator theoretic setting we obtain what today is called von Neumann's mean ergodic theorem (see Theorem 8.2 of [EFHN15]; see [vNe32] for the original result).

Theorem (Von Neumann's mean ergodic theorem)**.** *For a contraction* $T \in \mathscr{L}(H)$ *on a Hilbert space* H *the limit*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} T^n x$$

exists for every $x \in H$.

These first ergodic theorems were the starting point for a mathematical theory and its ramifications. Considering the problem from a functional analytic perspective has led to numerous new interesting problems. For example, mean ergodicity of operators on arbitrary Banach spaces has been studied extensively. Several characterizations are now available (see, e.g., Chapter 8 of [EFHN15]).

[5] Ergodicity is a notion of irreducibility of meaure-preserving systems, see, e.g. Chapter 3 of [Gla03] or Section 6.3 of [EFHN15].

Theorem. *For a power bounded operator $T \in \mathscr{L}(E)$ on a Banach space E and its Cesàro means $A_N := \frac{1}{N}\sum_{n=0}^{N-1} T^n$ for $N \in \mathbb{N}$ the following assertions are equivalent.*

(a) *T is mean ergodic, i.e., $\lim_{N\to\infty} A_N x$ exists for every $x \in E$.*

(b) *The sequence $(A_N x)_{N\in\mathbb{N}}$ has a weakly convergent subnet for every $x \in E$.*

(c) *The fixed space $\mathrm{fix}(T)$ separates the dual fixed space $\mathrm{fix}(T')$.*

(d) *$E = \mathrm{fix}(T) \oplus \overline{\mathrm{rg}}(I - T)$.*

(e) *There is an operator $P \in \mathscr{L}(E)$ such that $PT = TP = P$ and*

$$Px \in \overline{\mathrm{co}}\{T^n x \mid n \in \mathbb{N}_0\} \quad \text{for every } x \in E.$$

It is an immediate consequence of this result that every power-bounded operator on a reflexive Banach space is mean ergodic. For Banach spaces with a Schauder basis this property actually characterizes reflexivity (see [FLW01]).

The above theorem can be generalized in several ways to capture the asymtotic behaviour of different types of operators and operator families. For example, one can consider operators on locally convex spaces (see [ABR12] or [GK14]) or investigate— instead of working with a single operator $T \in \mathscr{L}(E)$—mean ergodic *semigroups* $\mathcal{S} \subseteq \mathscr{L}(E)$ (see [Nag73], [Sat78] and [Sch13]). In view of (b) of the theorem, it is clear that there is no difference between convergence of the Cesàro means with respect to the weak and strong operator topologies. However, one could ask for convergence with respect to the operator norm. This leads to uniformly mean ergodic operators (see, e.g., [Lin74] and Appendix W of [DNP87]). Based on Birkhoff's theorem a variety of ergodic theorems for operators on L^p-spaces has been proved, we mention the individual ergodic theorems for positive Dunford-Schwartz operators on $\mathrm{L}^1(X)$ (see [Hop54] and Theorem VIII.6.6 of [DS66]), for positive contractions on reflexive L^p-spaces (see [Akc75]), for vector-valued function spaces (see [BS57]) and the stochastic ergodic theorem for positive contractions on L^1-spaces (see Theorem 3.4.9 of [Kre85]).

Many recent results focus on so-called weighted and subsequential ergodic theorems (see Chapter 21 of [EFHN15] and the references given there), i.e., one is interested in convergence of weighted means

$$\frac{1}{N}\sum_{n=0}^{N-1} a_n T^n$$

where $(a_n)_{n\in\mathbb{N}}$ is a scalar sequence, or means of the form

$$\frac{1}{N}\sum_{n=0}^{N-1} T^{k_n}$$

where $(k_n)_{n\in\mathbb{N}}$ is a subsequence of the natural numbers. Returning to the physical motivation, this corresponds to giving more weight to measurements at certain points of time and missing measuring data at some points of time, respectively.

Finally, the investigation of convergence of multiple term Cesàro averages, e.g.,

$$\frac{1}{N}\sum_{n=0}^{N-1} T_\varphi^n f_1 \cdot T_\varphi^{2n} f_2 \cdots T_\varphi^{(k-1)n} f_{(k-1)}$$

for a measure-preserving system $(X;\varphi)$, $f_1, ..., f_{k-1} \in \mathrm{L}^\infty(X)$ and some $k \in \mathbb{N}$, has been of particular interest (see, e.g., [HK05], [Zie07] and [HK18]). The reason is the link between ergodic theory and number theory brought to light by Hillel Furstenberg in [Fur77]. It allows for an ergodic theoretic proof of the celebrated result of Endre Szemerédi (see [Sze75]) asserting that every subset of the natural numbers with positive upper density contains arithmetic progressions of arbitrary length.

Theorem. *Let $A \subseteq \mathbb{N}_0$ with upper density*

$$\overline{d}(A) := \limsup_{N\to\infty} \frac{|A \cap \{0, ..., N-1\}|}{N} > 0.$$

Then for each $k \in \mathbb{N}$ there is a pair $(a, n) \in \mathbb{N} \times \mathbb{N}$ with $a, a+n, a+2n, ..., a+(k-1)n \in A$.

To reformulate this as a problem of ergodic theory, consider the compact space $\{0,1\}^{\mathbb{N}_0}$ and the shift $\tau\colon \{0,1\}^{\mathbb{N}_0} \to \{0,1\}^{\mathbb{N}_0}$ defined by $\tau((x_n)_{n\in\mathbb{N}_0}) := (x_{n+1})_{n\in\mathbb{N}_0}$ for $(x_n)_{n\in\mathbb{N}_0} \in \{0,1\}^{\mathbb{N}_0}$. If A is a subset of $\mathbb{N}_0$, we can associate the characteristic function $\mathbb{1}_A \in \{0,1\}^{\mathbb{N}}$ with it and consider its closed orbit $K := \overline{\{\tau^n \mathbb{1}_A \mid n \in \mathbb{N}_0\}}$. The existence of an arithmetic progression in A of a fixed length $k \in \mathbb{N}$, i.e., a pair $(a, n) \in \mathbb{N} \times \mathbb{N}$ with $a, a+n, a+2n, ..., a+(k-1)n \in A$, is then equivalent to the existence of a number $n \in \mathbb{N}$ with

$$M \cap \tau^{-n}(M) \cap ... \cap \tau^{-(k-1)n}(M) \neq \emptyset \tag{1.1}$$

where $M := \{(x_n)_{n\in\mathbb{N}_0} \in K \mid x_0 = 1\}$. We have thus translated our number theoretic problem into the language of topological dynamics. By finding a suitable τ-invariant probability measure μ on K with $\mu(M) \geqslant \overline{d}(A)$ statement (1.1) becomes equivalent to

$$\limsup_{N\to\infty} \frac{1}{N}\sum_{n=0}^{N-1} \int \mathbb{1}_M \cdot T_\varphi^n \mathbb{1}_M \cdots T_\varphi^{(k-1)n} \mathbb{1}_M \,\mathrm{d}\mu > 0.$$

This observation reduces Szemerédi's theorem to a mutiple ergodic theorem for the Koopman operator and is part of Furstenberg's correspondence principle (see Theorem 20.13 of [EFHN15]).

Theorem. *The following two assertions are equivalent.*

(a) *For every measure-preserving system $(X;\varphi)$ and $0 < f \in \mathrm{L}^\infty(X)$*

$$\limsup_{N\to\infty} \frac{1}{N}\sum_{n=0}^{N-1} \int f \cdot T^n f \cdots T^{(k-1)n} f \,\mathrm{d}\mu > 0.$$

(b) *Every subset of the natural numbers with positive upper density contains arithmetic progressions of arbitrary length.*

This established a connection between ergodic theorems for the Koopman operator on one hand and theorems on structure in sets of positive integers on the other hand.

Dichotomy between structure and stability. The most structured systems appearing in topological dynamics are so-called *almost periodic* or *equicontinuous* in the sense that the powers $\{\varphi^n \mid n \in \mathbb{N}\}$ are an equicontinuous set of mappings. The standard examples for such structured systems are group rotations. Fix a compact group G and an element $a \in G$. Then $\varphi_a(g) := ag$ for $g \in G$ defines an almost periodic system $(G; \varphi_a)$.

Now assume that $(K; \varphi)$ is an almost periodic and invertible topological dynamical system. By the theorem of Arzelà-Ascoli it then follows that the closure of $\{T_\varphi^n \mid n \in \mathbb{Z}\}$ with respect to the strong operator topology is a compact topological group. Applying the Peter-Weyl theorem yields that $\mathrm{C}(K)$ is then the closed linear hull of the union of all eigenspaces $\ker(\lambda - T_\varphi)$ with $\lambda \in \mathbb{C}$ of modulus 1. A power-bounded operator $T \in \mathscr{L}(E)$ on a Banach space E with this property, i.e., such that

$$E = \overline{\mathrm{lin}}\,\{x \in E \mid \text{there is } \lambda \in \mathbb{C} \text{ with } |\lambda| = 1 \text{ and } Tx = \lambda x\},$$

is said to have *discrete spectrum*. We have thus seen that Koopman operators associated to almost periodic invertible topological dynamical systems always have discrete spectrum.[6] Topological and measure-preserving dynamical systems with discrete spectrum[7] which are irreducible are completely classified by the theorem of Halmos and von Neumann (see, e.g., Theorem 17.11 of [EFHN15] for the measure-preserving case; for further references and a generalization of the results to the non-irreducible case we refer to [Ede19]).

Theorem. (i) *A minimal topological dynamical system has discrete spectrum if and only if it is isomorphic to a group rotation of a compact monothetic*[8] *group. Moreover, two systems with discrete spectrum are isomorphic if and only if their Koopman operators have the same point spectrum.*

(ii) *An ergodic measure-preserving system has discrete spectrum if and only if it is isomorphic to a group rotation of a compact monothetic*[8] *group equipped with its Haar measure. Moreover, two systems with discrete spectrum are isomorphic if and only if their Koopman operators have the same point spectrum.*

This clearly shows the usefulness of spectral theory of the Koopman operator for the theory of dynamical systems. However, switching to the Koopman representation sometimes even allows for some sort of splitting between a structured and a stable part of the system. Recall that if T is a mean ergodic operator on a Banach space E, then the space E decomposes into a direct sum $\mathrm{fix}(T) \oplus \overline{\mathrm{rg}(I - T)}$. This is an important example for such a decomposition of a space into a

[6] Conversely, if T_φ has discrete spectrum, then $(K; \varphi)$ must be invertible and almost periodic.
[7] meaning that the induced Koopman operator has discrete spectrum
[8] A compact group G is *monothetic* if there is an element $a \in G$ such that $\{a^n \mid n \in \mathbb{Z}\}$ is a dense subgroup of G.

structured part (the fixed space $\operatorname{fix}(T)$) and a stable part where the powers T^n converge to zero in some sense (here, strongly in the mean). For contractions on Hilbert spaces there are a number of such decompositions (see, e.g., Lecture 6 of [EF]). A powerful result for operators on arbitrary Banach spaces is the decomposition theorem of Jacobs, de Leeuw and Glicksberg (see [Jac56], [LG61] and Chapter 16 of [EFHN15]).

Theorem. *If $T \in \mathscr{L}(E)$ is a bounded operator on a complex Banach space E with relatively weakly compact orbits, then $E = E_{\mathrm{rev}} \oplus E_{\mathrm{aws}}$ where the T-invariant closed subspaces E_{rev} and E_{aws} are given by*

$$E_{\mathrm{rev}} = \overline{\operatorname{lin}}\,\{x \in E \mid \text{there is a } \lambda \in \mathbb{C} \text{ with } |\lambda| = 1 \text{ and } Tx = \lambda x\},$$

$$E_{\mathrm{aws}} = \left\{ x \in E \,\middle|\, \lim_{N\to\infty} \sup_{\substack{x' \in E' \\ \|x'\| \leqslant 1}} \frac{1}{N} \sum_{n=0}^{N-1} |\langle T^n x, x' \rangle| = 0 \right\}.$$

In the situation of the theorem we can therefore decompose the space into a part on which T has discrete spectrum and a part where it is “stable” in some sense. This decomposition can be used to prove Roth’s theorem asserting that every subset $A \subseteq \mathbb{N}_0$ with positive upper density contains arithmetic progressions of length 3 (see Theorem 20.20 of [EFHN15]). A version of the decomposition results is also used in [MRR19] to show the sumset conjecture of Erdős.

Theorem. *Let $A \subseteq \mathbb{N}_0$ have positive upper density. Then there are infinite sets $B, C \subseteq \mathbb{N}_0$ such that $B + C \subseteq A$.*

For topological dynamical systems we obtain the Jacobs-de Leeuw-Glicksberg decomposition only in rare situations. For a measure-preserving system $(X; \varphi)$, however, the Koopman operator $T_\varphi \in \mathscr{L}(\mathrm{L}^p(X))$ always has relatively weakly compact orbits for $p \in [1, \infty)$. Since these operators are also mean ergodic, we already have two different decompositions of $\mathrm{L}^p(X)$ into a structured part and a stable part. In the proof of the convergence of the multiple means of the Koopman operator discussed above, the space $\mathrm{L}^2(X)$ is usually also decomposed into a structured part (a unital sublattice of $\mathrm{L}^2(X)$ corresponding to a so called *characteristic factor*) and a stable part where the multiple term Cesáro means go to zero. This decomposition can be obtained by defining certain seminorms (called *Gowers-Host-Kra-seminorms*) and decomposing $\mathrm{L}^2(X)$ into their kernels (the stable part) and the orthogonal complement (the structured part), see the last part of Section 20.3 of [EFHN15] for more details.

Factors, subsystems and topological models. Often, useful information of a dynamical system can be gained by investigating its subsystems and factors. While subsystems are (at least seemingly) easier to understand, it can be difficult to imagine the behavior of quotient systems. However, since assigning the corresponding functional analytic system to a topological or measure-preserving dynamical defines contravariant functors, factors of a topological or measure-preserving system correspond to subsystems of the induced Koopman system and vice versa. If $(K; \varphi)$ is a topological dynamical system, then its factors correspond precisely to the closed T_φ-invariant unital *-subalgebras of $\mathrm{C}(K)$. In particular, we can construct maximal factors with respect to some property by considering certain C^*-subalgebras of $\mathrm{C}(K)$ instead. For example, observing that the fixed space

$$A := \{f \in \mathrm{C}(K) \mid T_\varphi f = f\}$$

is a closed T_φ-invariant unital *-subalgebra of $\mathrm{C}(K)$ shows that $(K;\varphi)$ has a maximal factor with trivial dynamic which is unique up to isomorphy. Similar constructions can be done for measure-preserving systems $(X;\varphi)$ by considering closed T_φ-invariant sublattices of $\mathrm{L}^1(X)$ containing the unit $\mathbb{1}$.

The same idea can also be used to construct so-called *topological models* for measure-preserving systems relating topological dynamics and ergodic theory. Start with a measure-preserving system $(X;\varphi)$. As pointed out earlier, the Koopman operator also acts on $\mathrm{L}^\infty(X)$. Since $\mathrm{L}^\infty(X)$ is a commutative unital C^*-algebra and $T_\varphi \in \mathscr{L}(\mathrm{L}^\infty(X))$ is a unital $*$-homomorphism, we find a topological dynamical system $(K;\psi)$ representing $(\mathrm{L}^\infty(X);T_\varphi)$, i.e., there is a compact space K, a continuous mapping $\psi\colon K \to K$ and a $*$-isomorphism $V \in \mathscr{L}(\mathrm{C}(K),\mathrm{L}^\infty(X))$ such that $T_\varphi = VT_\psi V^{-1}$. Identifying the dual space $\mathrm{C}(K)'$ of $\mathrm{C}(K)$ with the space $\mathrm{M}(K)$ of complex regular Borel measures on K, we can consider V' as a mapping from $\mathrm{L}^1(X)$ (canonically embedded into $(\mathrm{L}^\infty(X))'$) to $\mathrm{M}(K)$ and obtain a ψ-invariant probability measure μ by setting $\mu := V'\mathbb{1}$ on K. The induced measure-preserving dynamical system on K is then isomorphic[9] to our original system and we have constructed a *topological model* for $(X;\varphi)$. The same procedure works if we replace $\mathrm{L}^\infty(X)$ by some closed $\mathrm{L}^1(X)$-dense T_φ-invariant unital *-subalgebra of $\mathrm{L}^\infty(X)$ (see Chapter 12 of [EFHN15] for the details).

Topological models are very useful since it is easier to work with a Borel measure on a compact space and a continuous dynamic (possibly with some additional properties) rather than with arbitary measure-preserving dynamics. The functional analytic apporach provides a systematic way to construct such topological models.

Let us return to the topological case once more. If $(K;\varphi)$ is a topological dynamical system, then for each closed subset $M \subseteq K$ the set $I_M := \{f \in \mathrm{C}(K) \mid f|_M = 0\}$ is a closed algebra and lattice ideal of $\mathrm{C}(K)$. Conversely, if I is such an ideal, then there is a unique closed subset $M \subseteq K$ with $I = I_M$. Moreover, M is φ-invariant if and only if I_M is T_φ-invariant. The subsystems of $(K;\varphi)$ are therefore encoded in the ideal structure of $\mathrm{C}(K)$.

This observation can be used for an operator theoretic approach to attractors (see [Kü19]). Here a non-empty closed and invariant set $M \subseteq K$ is an attractor if $\varphi^n(x)$ asymptotically approaches M for each $x \in K$. For the Koopman operator T_φ this corresponds to $\lim_{n\to\infty} T_\varphi^n = 0$ on I_M in some sense. For example, the sequence $\varphi^n(x)$ converges[10] to the set M if and only if T_φ restricted to I_M is strongly stable, i.e., $T_\varphi^n f \to 0$ for every $f \in I_M$. For operators on Banach spaces there is a natural hierarchy of stability concepts (see, e.g., [Eis10]) which can be related to different kinds of attractors. Consequently, attractors can be classified and studied using an operator theoretic approach.

[9] in some sense

[10] meaning that for each neighborhood U of M, the sequence is eventually contained in U

A plea for Koopmanism

Based on an extensive study of topological vector spaces (see, e.g., [Jar81] and [Sch99]), Banach lattices (see, e.g., [Sch74] and [MN91]) and operator algebras (see, e.g., [Dix77] and [Bla06]) functional analysis has become a powerful, deep and elegant theory combining algebra, topology and order theory. *Koopmanism* opens the door to apply linear functional analysis and all its descendants to dynamical systems. As indicated above, the Koopman operator has already proven to be useful for ergodic theory and topological dynamics at various occasions. The systematic operator theoretic approach to dynamical systems, however, is only at its beginning. Based on the manuscript [DNP87] and the book [EFHN15] this thesis contributes to this endeavor.

2 Objectives

In my Master studies I learned about the operator theoretic approach to topological dynamics and ergodic theory and wrote a thesis about compact operator semigroups (see [Kre16]). I then was eager to apply the tools from functional analysis, operator theory and topological algebra to dynamical systems. The main goal of my research was to extend the Koopman theory presented in [EFHN15]. More specifically, I focused on the structure theory of dynamical systems and examined qualitative properties such as mean ergodicity. The concrete topics which my research was concerned with and which are elaborated in this thesis have then be mostly motivated by ideas appearing in the *Arbeitsgemeinschaft Funktionalanalysis (AGFA)*, often brought to attention by my advisor Rainer Nagel.

A first step was to generalize the results of my Master's thesis from $\mathbb{N}_0$-actions to amenable semigroup actions and obtain a better understanding of compactifications of semigroups in locally convex topologies and their relations to mean ergodicity. This lead to my first article [Kre18].

Inspiration for my second project came from my courses in algebraic geometry and operator algebras. Here the spectra of rings or algebras (such as the maximal, prime or primitive spectrum) are used to study geometric objects (algebraic varieties or schemes) or to discuss representation theory (of C*-algebras or locally compact groups). Motivated by papers of Helmut Schaefer, I had the idea to consider a dynamical version of these concepts which finally led to [Kre19].

A further goal was to give an operator theoretic treatment of differentiable dynamical systems complementing the Koopman theory for topological dynamics and ergodic theory. Considering the derivatives of diffeomorphisms on the tangent bundle of some Riemannian manifold leads to the notions of Lyapunov coefficients, Sacker-Sell spetrum and exponential dichotomy. More generally, one can consider cocycles and so-called skew product flows. Sita Siewert and myself wanted to get a better understanding of these notions. This lead to an abstract setting for dynamics on Banach bundles and their operator theoretic description via dynamical Banach modules in [KS19].

In the last project of my time as a PhD student, Nikolai Edeko and myself returned to a problem that had been looming when I joined the AGFA several years ago. So called isometric and equicontinuous extensions play an important role in the structure theory of dynamical systems, in particular for Furstenberg's theorem on minimal and distal topological dynamical systems. We finally developed a deeper understanding of these structured extensions and their connections to groupoids, see [EK19].

3 Discussion of Results

As already indicated in the introduction, there are numerous ways to choose a mathematical setting for dynamical systems. For instance, one can consider differentiable dynamics on manifolds, actions of algebraic groups on algebraic varieties or complex dynamics on Riemannian surfaces. However, the results of this thesis are mostly concerned with topological and measure-preserving dynamical systems, i.e., group or semigroup actions on either a compact or locally compact space, or on a measure space. In this overview, we will focus on $\mathbb{N}_0$- and $\mathbb{Z}$-actions on compact spaces and probability spaces and refer to the attached articles for more general settings. We start by fixing our notation and recalling some basic concepts of topological dynamics and ergodic theory.

3.1 Preliminaries on dynamical systems

3.1.1 Topological dynamics and ergodic theory

The foundations of topological dynamics as a systematic theory were laid in the 1950s and can be found in the book of Gottschalk and Hedlund [GH55]. We refer to [Ell69], [Bro79], [Aus88], [dVr93], [Gla03] and [EE14] for classical treatments of this field. Here a *topological dynamical system* is a pair $(K;\varphi)$ such that K is a non-empty compact (Hausdorff) space and $\varphi\colon K \to K$ is a continuous map on that space. It is *invertible* if φ is a homeomorphism. A *morphism* $\vartheta\colon (K;\varphi) \to (L;\psi)$ between topological dynamical systems is given by a continuous mapping $\vartheta\colon K \to L$ which is compatible with the action, i.e., such that the diagram

$$\begin{tikzcd} K \arrow[r, "\varphi"] \arrow[d, "\vartheta"'] & K \arrow[d, "\vartheta"] \\ L \arrow[r, "\psi"'] & L \end{tikzcd}$$

commutes. It is called an *embedding* if the map ϑ is injective and an *extension* or a *factor mapping* if it is surjective. Finally, if $(K;\varphi)$ is a topological dynamical system, then a subset $L \subseteq K$ is φ*-invariant* if $\varphi(L) \subseteq L$. In this case $(L;\varphi|_L)$ is called a *subsystem* of $(K;\varphi)$.

We briefly list some simple and classical examples of (invertible) topological dynamical systems which will be used later on to illustrate the results.

Standard Examples. (i) Let $K := \mathbb{T} := \{z \in \mathbb{C} \mid |z| = 1\}$ be the circle and $a \in \mathbb{T}$. Then the rotation φ_a given by $\varphi_a(z) := az$ for $z \in \mathbb{T}$ defines a topological dynamical system $(\mathbb{T}; \varphi_a)$. More generally, every element a of a compact group G defines such a "group rotation".

(ii) Equip $K := \{0,1\}^{\mathbb{Z}}$ with the product topology. Then the shift φ given by $\varphi((x_n)_{n\in\mathbb{Z}}) := (x_{n+1})_{n\in\mathbb{Z}}$ for $(x_n)_{n\in\mathbb{Z}} \in K$ defines a topological dynamical system $(K; \varphi)$.

(iii) Consider the unit interval $K := [0,1]$. We obtain a dynamical system $(K; \varphi)$ by defining $\varphi(x) := x^2$ for every $x \in K$.

Of particular importance are dynamical systems which are irreducible in some sense. The following natural notions of "irreducibility" will be of relevance for our discussion.

Definition. A topological dynamical system $(K; \varphi)$ is

(i) *minimal* if it has no non-trivial subsystems.

(ii) *(topologically) transitive* if $K = \overline{\mathrm{orb}}(x) := \overline{\{\varphi^n(x) \mid n \in \mathbb{N}_0\}}$ for some $x \in K$.

(iii) *(topologically) ergodic* if every factor of $(K; \varphi)$ with trivial dynamic is trivial, i.e., its underlying space is a singleton.

Given a topological dynamical system $(K; \varphi)$, we call a closed φ-invariant subset $M \subseteq K$ *minimal*, if the correspoding subsystem $(M; \varphi|_M)$ is minimal. Note that if $(K; \varphi)$ is invertible, then a closed φ-invariant subset $M \subseteq K$ is minimal if and only if it is minimal among invertible systems, i.e., M is also φ^{-1}-invariant and the invertible system $(M; \varphi|_M)$ has no non-trivial invertible subsystems (see Remarks 3.2 of [EFHN15]).

It is easy to check that in the Definition above (i) implies (ii) which in turn implies (iii). Returning to our standard examples, we see that example (i) is minimal if a is not a root of unity and not even ergodic otherwise. Moreover, (ii) is transitive, but not minimal, and one can show that (iii) is ergodic and non-transitive.

We now turn to the measurable case and ergodic theory (see [Bro70], [Wal75], [Par81], [Kre85], [EW11] and [VO16] for an introduction). A *measure-preserving system* $(X; \varphi)$ consists of a probability space $X = (\Omega_X, \Sigma_X, \mu_X)$ and a measurable mapping $\varphi\colon \Omega_X \to \Omega_X$ which is *measure-preserving*, i.e, $\mu_X(\varphi^{-1}(A)) = \mu_X(A)$ for every $A \in \Sigma_X$. It is *invertible* if φ is *essentially invertible*, i.e., there is a measurable mapping $\sigma\colon \Omega_X \to \Omega_X$ such that $\varphi \circ \sigma = \mathrm{id}_{\Omega_X} = \sigma \circ \varphi$ almost everywhere.

If $(K; \varphi)$ is a topological dynamical system, then by the Theorem of Krylov-Bogolyubov (see, e.g., Theorem 4.1 of [Gla03]), there always is a regular Borel probability measure μ on K which is invariant under φ and therefore induces a measure-preserving system which we denote by $(K, \mu; \varphi)$. For example, the Haar measure of $\mathbb{T}$ defines an invariant probability measure for Standard Example (i). The product measure of $\frac{1}{2}(\delta_0 + \delta_1)$ on $\{0,1\}^{\mathbb{Z}}$ is an invariant measure for Standard Example (ii). The resulting system is a *two-sided Bernoulli shift*. Finally, in our third example (iii), all invariant measures are convex combinations of the two Dirac measures δ_0 and δ_1.

The "irreducible" measure-preserving systems are called *ergodic*. Ergodicity in this case can be defined in several ways (see Section 6.3 of [EFHN15]), e.g., every $A \in \Sigma(X)$ with $\varphi^{-1}(A) \subseteq A$ has measure zero or one. Given a topological dynamical system $(K;\varphi)$, an invariant probability measure μ is called *ergodic* if the induced system $(K,\mu;\varphi)$ is ergodic. If a system $(K;\varphi)$ admits only one invariant probability measure, then this has to be ergodic. Such systems are called *uniquely ergodic* and are automatically minimal. Standard Example (i) is such a system if a is not a root of unity.

3.1.2 The operator theoretic approach

We consider topological and measure-preserving dynamical systems using Koopmanism. A systematic introduction to this operator theoretic approach can be found in [EFHN15]. Since the benefits and applications of the Koopman philosophy have already been discussed in the introduction, we only recall the notation here. For a compact space K we denote by $\mathrm{C}(K)$ the space of all complex[1]-valued continuous functions on K and identify its dual space $\mathrm{C}(K)'$ with the space of all complex regular Borel measures on K. If $(K;\varphi)$ is a topological dynamical system we define its *Koopman operator* $T_\varphi \in \mathscr{L}(\mathrm{C}(K))$ by $T_\varphi f := f \circ \varphi$ for each $f \in \mathrm{C}(K)$. The mapping T_φ is a bounded operator on $\mathrm{C}(K)$ which is also a lattice and *-homomorphism preserving the unit $\mathbb{1} \in \mathrm{C}(K)$.

For a measure space $X = (\Omega_X, \Sigma_X, \mu_X)$ we write $\mathrm{L}^p(X)$ for the corresponding complex L^p-spaces where $p \in [1,\infty]$. If $(X;\varphi)$ is a measure-preserving system, we define its Koopman operator $T_\varphi f := f \circ \varphi$ for every measurable function $f \colon \Omega_X \to \mathbb{C}$. This defines a linear mapping on the vector space of all measurable complex valued functions which induces an operator on the Banach spaces $\mathrm{L}^p(X)$ for every $p \in [1,\infty]$, also denoted by T_φ. For $p \neq \infty$ this is an isometry and a bi-Markov lattice homomorphism, i.e., $|T_\varphi f| = T_\varphi|f|$ for all $f \in \mathrm{L}^p(X)$, $T_\varphi \mathbb{1} = \mathbb{1}$ and $T_\varphi' \mathbb{1} = \mathbb{1}$.

We point out that some irreducibility notions of dynamical systems can be characterized nicely using Koopman operators. A topological dynamical system $(K;\varphi)$ is minimal if and only if the Koopman operator T_φ is irreducible in the sense that there are no non-trivial closed T_φ-invariant ideals in $\mathrm{C}(K)$ (see Corollary 4.9 of [EFHN15]) and it is ergodic if an only if T_φ has a one-dimensional fixed space in $\mathrm{C}(K)$ (an easy consequence of the Gelfand representation theorem). Similarly, a measure-preserving system $(X;\varphi)$ is ergodic if and only if there are no non-trivial closed T_φ-invariant ideals in $\mathrm{L}^p(X)$ for one/every $p \in [1,\infty)$ which is the case if and only if T_φ has a one-dimensional fixed space in all these spaces (see Proposition 7.15 of [EFHN15]).

[1] Some results discussed below also hold in the real case.

3.2 Enveloping semigroups

3.2.1 Enveloping semigroups in topological dynamics and operator theory

The idea to associate an enveloping semigroup to group actions on compact spaces goes back to Robert Ellis, see [EG60] and [Ell60]. Given an invertible topological dynamical system $(K;\varphi)$ we can consider its *Ellis semigroup* defined by

$$\mathrm{E}(K;\varphi) := \overline{\{\varphi^n \mid n \in \mathbb{Z}\}} \subseteq K^K$$

where the closure is taken with respect to the product topology. Clearly, $\mathrm{E}(K;\varphi)$ is a compact space. Moreover, with the composition of mappings as multiplication this is in fact a semigroup. It is *right topological* in the sense that $\vartheta \mapsto \vartheta \circ \varrho$ is continuous for each fixed $\varrho \in \mathrm{E}(K;\varphi)$. For such compact right topological semigroups there is a deep structure theory (see [BJM78] and [BJM89]) which can now be employed to study topological dynamical systems. It turns out that many properties of the system $(K;\varphi)$ can be characterized in terms of algebraic or topological properties of $\mathrm{E}(K;\varphi)$ (see, e.g., Proposition 2.5 and Theorem 3.1 of [Gla07a]).
The idea of compact enveloping semigroups has also been applied to operator theory. For operators with relatively compact orbits this leads to enveloping operator semigroups of Jacobs, de Leeuw and Glicksberg. Applying the structure theory of semitopological semigroups leads to the famous Jacobs-deLeeuw-Gliecksberg-decomposition stated in the introduction. Witz (see [Wit64]) and later Köhler ([Kö94] and [Kö95]) introduced an enveloping semigroup for arbitary power bounded operators $T \in \mathscr{L}(E)$ (or even bounded semigroups) on a Banach space E by taking the closure of $\{((T')^n \mid n \in \mathbb{N}_0\} \subseteq \mathscr{L}(E')$ with respect to the weak* operator topology.
In [Kre18] a general and systematic approach to enveloping operator semigroups is pursued. Given a semigroup $\mathcal{S}$ of continuous operators on any locally convex space X we introduce its *Köhler semigroup* as the closure $\mathcal{K}(\mathcal{S}) := \overline{\mathcal{S}} \subseteq X^X$ with respect to the product topology. In Section 2 and 3 of [Kre18] properties of these operator semigroups are then discussed and applied to the different enveloping semigroups appearing in topological dynamics.

3.2.2 Convex enveloping semigroups and mean ergodicity

A power-bounded operator $T \in \mathscr{L}(E)$ on a Banach space E is said to be *(*strongly) mean ergodic if the limit $P := \lim_{N\to\infty} \frac{1}{N}\sum_{n=0}^{N-1} T^n$ exists with respect to the strong (or—equivalently—the weak) operator topology. In this case the operator $P \in \mathscr{L}(E)$ (called the *mean ergodic projection*) is contained in $\mathcal{J}_{\mathrm{c}}(T) := \mathcal{K}(\mathrm{co}\,\{T^n \mid n \in \mathbb{N}_0\})$ where the Köhler semigroup is taken with respect to the norm topology of E. Moreover, P is a zero element of $\mathcal{J}_{\mathrm{c}}(T)$ i.e., $PS = SP = P$ for every $S \in \mathcal{J}_{\mathrm{c}}(T)$. It turns out that the existence of such a zero element is actually equivalent to mean ergodicity (see [Nag73] and [Sch13]). This characterization shows that there is no need to define mean ergodicity via convergence of the Cesàro means. If T is mean ergodic then every net $(T_\alpha)_{\alpha\in A}$ in $\mathcal{J}_{\mathrm{c}}(T)$ such that

$$(I - T)T_\alpha = 0$$

in the strong operator topology, converges strongly to P. Conversely, if any of these so called *ergodic operator nets* converges, then T is mean ergodic.
The situation changes when we pass to a dual notion of mean ergodicity as considered by A. Romanov in [Rom11] and [Rom16] (and partly earlier by Day in [Day50] and Witz in [Wit64]). Let us call T *weak* mean ergodic* if all weak* ergodic operator nets converge in the weak* operator topology. Then T is weak* mean ergodic if and only the convex Köhler semigroup $\mathcal{K}_c(T') := \mathcal{K}(\mathrm{co}\,\{(T')^n \mid n \in \mathbb{N}_0\})$ with respect to the weak* topology on X' contains a zero element (see Theorem 1.3 of [Rom11]). However, in contrast to strong mean ergodicity, the convergence of one ergodic operator net is not sufficient for weak* mean ergodicity (see Example 5.14 of [Kre18]).
In [Kre18] this characterization is generalized to operators and semigroups on more general locally convex spaces (see Theorem 4.3 of [Kre18]). In particular, strong and weak* mean ergodicity are characterized for operators and semigroups on barrelled locally convex spaces (see Theorem 4.7 of [Kre18]).
In the special case of Koopman operators we now have three notions of ergodicity: Weak* mean ergodicity, strong mean ergodicity and unique ergodicity. Returning to our standard examples, we see that (i) is always strongly mean ergodic and uniquely ergodic if and only if a is not a root of unity. Example (ii) is not even weak* mean ergodic, while it can be shown that (iii) is weak* mean ergodic but not strongly mean ergodic.
In [Kre18] we obtain a description of all these notions in terms of zero elements of the convex Köhler semigroup $\mathcal{K}_c(K;\varphi) := \mathcal{K}_c(T_\varphi')$ as summarized by the following result.

Theorem. *For a topological dynamical system $(K;\varphi)$ the following assertions hold.*

(a) *$(K;\varphi)$ is weak* mean ergodic if and only if $\mathcal{K}_c(K;\varphi)$ has a zero.*

(b) *$(K;\varphi)$ is strongly mean ergodic if and only if $\mathcal{K}_c(K;\varphi)$ has a zero which is a weak* continuous operator.*

(c) *$(K;\varphi)$ is uniquely ergodic if and only if $\mathcal{K}_c(K;\varphi)$ has a zero which is a rank one operator.*

Moreover, all these notions are equivalent if $(K;\varphi)$ is topologically transitive.

3.2.3 Tame dynamical systems

Some notions of structuredness of a topological dynamical system $(K;\varphi)$ can be described by compactness properties of the orbits of the Koopman operator on one hand, and by topological or algebraic properties of the Ellis semigroup $\mathrm{E}(K;\varphi)$ on the other hand. For example, an invertible system $(K;\varphi)$ is equicontinuous, i.e., $\{\varphi^n \mid n \in \mathbb{Z}\}$ is an equicontinuous set of mappings, if and only if T_φ has relatively norm compact orbits $\{T_\varphi^n f \mid n \in \mathbb{N}_0\}$ for $f \in \mathrm{C}(K)$. This is the case if and only if the Ellis semigroup $\mathrm{E}(K;\varphi)$ is a topological group consisting of continuous mappings (see Theorem 1.8 of [Gla03]). We obtain a new notion of structuredness by requiring a weaker compactness property for the orbits of the Koopman operator.

Definition 1. A metric topological dynamical system $(K;\varphi)$ is *tame* if T_φ has relatively sequentially compact orbits with respect to the product topology of $\mathbb{C}^K$, i.e., for $f \in \mathrm{C}(K)$, every sequence in $\{T_\varphi^n f \mid n \in \mathbb{N}_0\}$ has a pointwise convergent subsequence.

Tame dynamical systems have been introduced (under a different name) by A. Köhler in [Kö94] and [Kö95] and have become an important class of dynamical systems (see, e.g., [Gla06], [Gla07b], [KL07], [GM15], [Gla18] and Chapter 8 of [KL16]). Once again, the enveloping semigroups of a metric topological dynamical system $(K;\varphi)$ can be used to characterize tameness of $(K;\varphi)$, see Proposition 3.11 and Remark 3.12 of [Kre18]. In our standard examples, (i) is almost periodic and, in particular, tame, while (ii) is not tame and (iii) is tame but not almost periodic.
Tame systems have the benefit that the topology of its enveloping semigroups can be largely descibed by sequences instead of nets. This can be used to obtain a simple proof of the following result of Glasner (see Theorem 5.1 of [Gla07b] for abelian group actions and Corollary 5.4 in [Kre18] for actions of amenable semigroups).

Theorem. *Every metric topological dynamical system, which is minimal and tame, is uniquely ergodic.*

In addition, we obtain a nice characterization of weak* mean ergodicity for such systems (see Theorem 5.10 of [Kre18]).

Theorem. *A metric topological dynamical system $(K;\varphi)$, which is tame, is weak* mean ergodic if and only if every closed orbit $\overline{\mathrm{orb}}(x)$ contains a unique minimal set.*

This provides a simple way to see that Standard Example (iii) is in fact weak* mean ergodic.

3.3 Primitive spectrum

3.3.1 Spectra for C*-algebras

The famous Gelfand theory allows to represent every commutative unital C*-algebra A as a space $\mathrm{C}(K)$. The compact space K can be chosen to be the *Gelfand space* $\mathrm{X}(A)$ consisting of all non-zero multiplicative linear functionals on A equipped with the weak* topology. Another approach is to consider the set $\mathrm{Max}(A)$ of all maximal ideals of A—which are precisely the kernels of non-zero multiplicative functionals—and equip this set with the so-called Jacobson or Zariski topology. This is called the *maximal spectrum* of A.
In the non-commutative case one can still consider maximal two-sided ideals, but the so called *primitive spectrum* $\mathrm{Prim}(A)$ of a C*-algebra A contains more information and has broader applications in its representation theory (see, e.g., Chapter 3 of [Dix77] or Section II.6.5 of [Bla06]). Here an ideal p of A is called *primitive* if it is the kernel of a non-zero irreducible unitary representation of A as operators on a Hilbert space. Again one can equip $\mathrm{Prim}(A)$ with the Jacobson topology to obtain a quasi-compact but generally non-Hausdorff space. Moreover,

this agrees with the maximal spectrum if A is commutative. A result of Dauns and Hofmann (see, e.g., [Hof11] or Theorem II.6.5.10. of [Bla06]) shows that if A is unital, the space of continuous functions on $\mathrm{Prim}(A)$ is canonically isomorphic to the center of A.
Similar concepts are used with great effect in algebraic geometry in the context of affine algebras or commutative rings to construct affine varieties and affine schemes (see, e.g., [GW10]).

3.3.2 The primitive spectrum of a Markov operator

We now introduce a dynamical version of the primitive spectrum. Consider a Markov operator S on a space $\mathrm{C}(K)$, i.e., a positive operator preserving the one function $\mathbb{1} \in \mathrm{C}(K)$. Proper closed ideals $I \subseteq \mathrm{C}(K)$ which are S-invariant, i.e., $SI \subseteq I$, are called *S-ideals*[2]. If $S = T_\varphi$ is the Koopman operator of a dynamical system $(K;\varphi)$, then these are of the form

$$I_L := \{f \in \mathrm{C}(K) \mid f|_L = 0\}$$

where $L \subseteq K$ is a non-empty and closed φ-invariant subset of K. We call L the *support* of the ideal I_L. In two articles H. H. Schaefer used the S-ideal structure to examine the spectral and ergodic properties of S (see [Sch67] and [Sch68]). Inspired by his work and by courses in algebraic geometry and on operator algebras I asked if one could in this context find a natural analogon of primitive ideals. Replacing the notion of irreducible representations by ergodic invariant measures and kernels of these representations by absolute kernels of the ergodic measures[3] leads to the following (see Definition 2.6 of [Kre19]).

Definition. Let S be a Markov operator on $\mathrm{C}(K)$. An S-ideal $p \subseteq \mathrm{C}(K)$ is *primitive* if it is the absolute kernel of an ergodic measure, i.e.,

$$p = I_\mu := \{f \in \mathrm{C}(K) \mid \langle |f|, \mu\rangle = 0\}$$

for some ergodic measure μ on K.

One can show that every maximal S-ideal (with respect to inclusion) is primitive (see Corollary 2.10 of [Kre19]), but the converse generally does not hold (see Example 2.11 of [Kre19]). The set $\mathrm{Prim}(S)$ of all primitive S-ideals can now be topologized as in the case of C*-algebras using the notions of hull and kernel (see Definition 3.1 of [Kre19]).

Definition. Let S be a Markov operator on $\mathrm{C}(K)$.

(i) For a subset $I \subseteq \mathrm{C}(K)$ we define its *hull* by

$$\mathrm{hull}(I) := \{p \in \mathrm{Prim}(S) \mid I \subseteq p\}.$$

[2] More generally, we could consider ideals which are invariant under an amenable semigroup of Markov operators here (cf. [Kre19]), but we confine ourselves to the case of a single operator in this discussion.

[3] The definitions of invariant and ergodic measures as well as the concepts of unique and mean ergodicity discussed in the previous sections can be extended to Markov operators on $\mathrm{C}(K)$ in a straightforward way.

(ii) For a subset $A \subseteq \mathrm{Prim}(S)$ we define its *kernel* by

$$\ker(A) := \bigcap_{p \in A} p \subseteq \mathrm{C}(K).$$

The mapping $A \mapsto \overline{A} := \mathrm{hull}(\ker(A))$ then defines a Kuratowski closure operator and thereby a topology on $\mathrm{Prim}(S)$. The topological space $\mathrm{Prim}(S)$ has similar properties as the primitive spectrum of non-commutative C*-algebras or the topology of affine varieties and schemes of algebraic geometry (see Proposition 5.5 of [Kre19]). In particular, it is non-Hausdorff in general. A more intuitively accessible description of the primitive spectrum can be obtained by identifying a primitive ideal with the support of the corresponding ergodic measure (which is possible, since these supports and the primitive ideals are in one to one correspondence). A net of supports $(\mathrm{supp}\,\mu_i)_{i\in I}$ of ergodic measures then converges to a support $\mathrm{supp}\,\mu$ if and only if for every open subset $U \subseteq K$ having non-empty intersection with $\mathrm{supp}\,\mu$ there is $i_0 \in I$ such that $U \cap \mathrm{supp}\,\mu_i \neq \emptyset$ for every $i \geqslant i_0$.
Using this perspective and returning to the standard examples we see that the primitive spectrum of (i) consists of one point if a is not a root of unity since the system is minimal in this case. If $a^n = 1$ then the primitive spectrum consists of the (ideals corresponding to) all orbits $\overline{\mathrm{orb}}(b) = \{a^k b \mid k \in \{0, \ldots, n-1\}$ where $b \in \mathbb{T}$. In example (iii) the primitive spectrum consists of (the ideals defined by) $\{0\}$ and $\{1\}$. In example (ii) the primitive spectrum is quite large and non-Hausdorff. For example, the elements of $\mathrm{Prim}(T_\varphi)$ defined by the fixed points $(0)_{n\in\mathbb{Z}}$ and $(1)_{n\in\mathbb{Z}}$ cannot be separated by open sets (cf. Example 5.7 (iii) of [Kre19]).

3.3.3 Radical ideals and centers of attraction

In case of the prime spectrum in algebraic geometry the so-called *radical ideals* are precisely the intersections of prime ideals. In our situation an *S-ideal I* is *radical* if $I = \mathrm{rad}_S(I) := \ker(\mathrm{hull}(I))$[4]. It turns out that in our setting the radical ideals are related to stability properties of the operator S (see Theorem 4.2 and Corollary 4.3 of [Kre19]).

Theorem. *Let S be a Markov operator on* $\mathrm{C}(K)$*. For each S-ideal* $I = I_L \subseteq \mathrm{C}(K)$ *we have*

$$\begin{aligned}
\mathrm{rad}_S(I) &= \left\{ f \in \mathrm{C}(K) \,\middle|\, \lim_{N\to\infty} \left(\frac{1}{N} \sum_{n=0}^{N-1} S^n |f| \right)|_L = 0 \textit{ uniformly} \right\} \\
&= \left\{ f \in \mathrm{C}(K) \,\middle|\, \lim_{N\to\infty} \left(\frac{1}{N} \sum_{n=0}^{N-1} S^n |f| \right)|_L = 0 \textit{ weakly in } \mathrm{C}(L) \right\} \\
&= \left\{ f \in \mathrm{C}(K) \,\middle|\, \lim_{N\to\infty} \left(\frac{1}{N} \sum_{n=0}^{N-1} S^n |f| \right)|_L = 0 \textit{ pointwise} \right\}.
\end{aligned}$$

[4]In [Sch68] Schaefer defined the radical (of an operator S) as the intersection of all maximal S-ideals.

In the special case of a Koopman operator this leads to the characterization of so-called minimal centers of attraction. Given a topological dynamical system $(K;\varphi)$ a non-empty closed φ-invariant subset L of K is a *center of attraction* (see, e.g., [Sig77] or Exercise I.8.3 of [Man87]) if

$$\lim_{N\to\infty} \frac{|\{n \in \{0,\dots,N-1\} \mid \varphi^n(x) \in U\}|}{N} = 1$$

for every neighborhood U of L and every $x \in K$. This type of attraction is quite weak since the orbit of a point can move far away from L as long as it returns to any neighborhood of L "often enough". We can now characterize these centers of atttraction in terms of radical ideals.

Theorem. *Let $(K;\varphi)$ be a topological dynamical system. Then a non-empty closed invariant subset $L \subseteq K$ is a center of attraction if and only if $I_L \subseteq \mathrm{rad}_S(0)$. In particular, there always is a smallest minimal center of attraction $\mathrm{M}(S)$ given by the support of $\mathrm{rad}_S(0)$.*

What are the minimal centers of attraction in our standard examples? The space in (i) is always the union of minimal sets which implies that the minimal center of attraction is the whole space. In (ii) the union of all orbits of periodic points are dense which again implies that K is the minimal center of attraction. In example (iii) it is the set $\{0,1\} \subseteq [0,1]$.

3.3.4 Representation of the fixed space and mean ergodicity

Let us return to a general Markov operator S on $\mathrm{C}(K)$. As said above, the Dauns-Hofmann-theorem asserts that for a unital C*-algebra A the space of continuous functions on the primitive spectrum of A is canonically isomorphic to the center of A. What happens in the case of the "dynamical primitive spectrum"? It turns out that the space $\mathrm{C}(\mathrm{Prim}(S))$ is canonically isomorphic to the fixed space of the operator induced by S on $\mathrm{C}(\mathrm{M}(S))$ (see Theorem 6.5 of [Kre19]). This can be applied to give a characterization of S to be (strongly) mean ergodic (see Theorem 7.1 of [Kre19]).

Theorem. *For a Markov operator $S \in \mathscr{L}(\mathrm{C}(K))$ the following assertions are equivalent.*

(a) *The operator S is mean ergodic.*

(b) *The following three conditions are satisfied.*

   (i) *The primitive spectrum $\mathrm{Prim}(S)$ is a Hausdorff space.*

   (ii) *For every ergodic measure μ the support $\mathrm{supp}\,\mu$ is uniquely ergodic.*

   (iii) *Every fixed function in $\mathrm{C}(\mathrm{M}(S))$ has an invariant continuous extension to K.*

If S is radical free, i.e., $\mathrm{rad}_S(0)$ is the zero ideal, then clearly (iii) of (b) always holds. Coming back to our standard examples we can employ the characterization to see (again!) that (i) is mean ergodic. Example (ii) is not mean ergodic since the primitive spectrum is not a Hausdorff space. Finally, Example (iii) is not mean ergodic since the invariant continuous function $f\colon \{0,1\} \to \mathbb{C}$ defined by $f(0) := 0$ and $f(1) := 1$ has no continuous invariant extension to the whole space

$K = [0, 1]$. We refer to Example 7.5 of [Kre19] for more examples showing that the conditions (i) - (iii) above are independent.

3.4 Dynamical Banach bundles and modules

3.4.1 Dynamics on Banach bundles

In the previous sections we have been concerned with topological dynamics and their Koopman linearizations. However, if we assume that we have a *diffeomorphism* of a Riemannian manifold (as in Standard Example (i)), the Koopman operator on the space of continuous functions clearly does not reflect all available geometric information. One could now pass to different Banach function spaces and consider Koopman linearizations on spaces of differentiable functions. In many contexts, however, one is interested in the derivative of a flow (or other cocycles) acting on the tangent bundle (see, e.g., Anosov flows and hyperbolic dynamical systems in Chapter 1 of [BP13]). Abstractly speaking, this is an example of dynamics on a bundle of (finite-dimensional) Banach spaces. Such "dynamical Banach bundles" also appear in other contexts. Given an extension $q: (K; \varphi) \to (L; \psi)$ of invertible topological dynamical systems we can decompose the space K into its fibers. This induces a fibering of the corresponding space $\mathrm{C}(K)$ into a bundle of Banach spaces and φ induces a dynamic on this bundle (see Example 2.3 of [KS19]).
These two examples look quite differently and we strived to introduce an abstract setting including both. This leads to so called "dynamical Banach bundles $(E; \Phi)$ over an invertible dynamical system $(K; \varphi)$"[5] (see Definitions 2.1, 2.6 and 2.7 in [KS19] for the precise definition). In [KS19] we present a systematic investigation by means of operator theory.

3.4.2 Weighted Koopman operators

Given a dynamical Banach bundle $(E; \Phi)$ over an invertible dynamical system $(K; \varphi)$ we consider the Banach space $\Gamma(E)$ of continuous sections of E equipped with the supremum norm. On this space we obtain the *weighted Koopman operator* $\mathcal{T}_\Phi$ induced by Φ by $\mathcal{T}_\Phi s := \Phi \circ s \circ \varphi^{-1}$ for $s \in \Gamma(E)$. In case of the derivative of a diffeomorphism on a tangent bundle of a Riemannian manifold this is just the push forward of vector fields.
We tried to give an abstract characterization of such weighted Koopman operators in terms of algebraic or lattice theoretic properties in analogy to (non-weighted) Koopman operators on $\mathrm{C}(K)$. More precisely, given a fixed invertible topological dynamical system $(K; \varphi)$, a Banach bundle E over K and a bounded operator $\mathcal{T} \in \mathscr{L}(\Gamma(E))$ can we characterize when $\mathcal{T}$ is a weighted Koopman operator?
To answer this question we first have to elaborate the algebraic and lattice theoretic structure of the space $\Gamma(E)$. Observe that $\Gamma(E)$ is canonically a module over $\mathrm{C}(K)$ and in fact a *Banach module*, i.e., $\|fs\| \leqslant \|f\| \cdot \|s\|$ for all $f \in \mathrm{C}(K)$ and $s \in \Gamma(E)$. On the other hand, we obtain

[5] The results of this section extend to dynamics on locally compact spaces and σ-finite measure spaces. For the sake of simplicity we remain in the framework introduced in Section 3.1.1.

a "vector-valued norm" $|\cdot|\colon \Gamma(E) \to \mathrm{U}(K)_+$, where $\mathrm{U}(K)_+$ is the space of positive upper semicontinuous functions on K, by setting $|s|(x) := \|s(x)\|$ for $x \in K$ and $s \in \Gamma(E)$. Indeed, this mapping satisfies the following properties for all $s, t \in \Gamma(E)$ and $f \in \mathrm{C}(K)$.

(i) $\|s\| = \||s|\|_\infty$.

(ii) $|fs| = |f| \cdot |s|$.

(iii) $|s + t| \leqslant |s| + |t|$.

It turns out that weighted Koopman operators can now be characterized either via the module structure or via the "vector valued norm" (see Theorem 5.5 of [KS19]).

Theorem. *Let $(K; \varphi)$ be an invertible topological dynamical system, E a Banach bundle over K and T_φ the corresponding Koopman operator on $\mathrm{C}(K)$ and $\mathrm{U}(K)_+$, respectively. For a bounded operator $\mathcal{T} \in \mathscr{L}(\Gamma(E))$ the following assertions are equivalent.*

(a) *$\mathcal{T} = \mathcal{T}_\Phi$ for some dynamical Banach bundle $(E; \Phi)$ over $(K; \varphi)$.*

(b) *$\mathcal{T}$ is a T_φ^{-1}-*homomorphism*, i.e., $\mathcal{T}(fs) = T_\varphi^{-1} f \cdot \mathcal{T} s$ for every $f \in \mathrm{C}(K)$ and $s \in \Gamma(E)$.*

(c) *$\mathcal{T}$ is T_φ^{-1}-*dominated*, i.e., $|\mathcal{T} s| \leqslant \|\mathcal{T}\| \cdot T_\varphi^{-1}|s|$ for every $s \in \Gamma(E)$.*

What can be said about the measurable case? Similar to the topological setting we define the notion of dynamical measurable Banach bundles $(E; \Phi)$ over invertible measure-preserving systems $(X; \varphi)$. and consider the induced weighted Koopman operators on spaces $\Gamma^1(E)$ of (equivalence classes of) integrable sections. These spaces $\Gamma^1(E)$ are then Banach modules over $\mathrm{L}^\infty(X)$ and admit an "$\mathrm{L}^1(X)$-valued norm" $|\cdot|\colon \Gamma^1(E) \to \mathrm{L}^1(X)$. Replacing the objects accordingly, an analogous characterization of weighted Koopman operators also holds in the measurable setting under some separability assumptions (see Theorem 5.16 of [KS19]).

3.4.3 Gelfand-type theorems

The classical theorems of Gelfand and Kakutani, respectively, represent abstract commutative unital C*-algebras and AM-spaces with order unit as spaces $\mathrm{C}(K)$ for some compact space K and AL-spaces as $\mathrm{L}^1(X)$ for a measure space X (see, e.g. Theorem 4.23 of [EFHN15] and Sections II.7 and II.8 of [Sch74]). In the 70s and 80s of the past century a representation theory for Banach modules has also been developed (see, e.g., [HK77] or Chapter 2 of [DG83]). The well-known Serre-Swan theorem (see [Swa62]) establishes a connection between vector bundles over a compact space K and finitely generated and projective modules over the algebra $\mathrm{C}(K)$. Therefore, one might expect that certain Banach modules are isomorphic to section spaces of Banach bundles. In fact, a Banach module Γ over $\mathrm{C}(K)$ is isometrically isomorphic to a space $\Gamma(E)$ if and only if it is *(locally)* $\mathrm{C}(K)$*-convex* meaning that $\|fs + gt\| = \max(\|s\|, \|t\|)$ for all $s, t \in \Gamma$ and positive functions $f, g \in \mathrm{C}(K)$ with $f + g = \mathbb{1}$. An equivalent condition is the following: Whenever $f, g \in \mathrm{C}(K)$ are positive and $s \in \Gamma$, we have $\|(f + g)s\| = \|fs\| + \|gs\|$ (see Section 7 of [Gie82]).
Now, given a Banach module Γ over $\mathrm{C}(K)$ we can consider the closed submodule $\Gamma_s := \overline{\mathrm{C}(K)s}$

generated by an element $s \in \Gamma$. It can be shown that this can be canonically turned into a Banach lattice (see Lemma 4.6 of [AAK92] and Proposition 4.1 of [KS19]). The condition above then actually means that Γ_s is an AM-space for each $s \in \Gamma$. Moreover, a result of Cunningham (see [Cun67] and Proposition 4.17 of [KS19]) shows that there is a duality between Banach modules satisfying this AM-condition and Banach modules satisfying an analogous AL-condition:

Proposition. *For a Banach module Γ over $\mathrm{C}(K)$ and its dual space Γ' equipped with the canonical structure of a Banach module over $\mathrm{C}(K)$ the following assertions hold.*

(i) Γ_s *is an AM-space for each $s \in \Gamma$ if and only if $(\Gamma')_{s'}$ is an AL-space for each $s' \in \Gamma'$.*

(ii) Γ_s *is an AL-space for each $s \in \Gamma$ if and only if $(\Gamma')_{s'}$ is an AM-space for each $s' \in \Gamma'$.*

This duality inspired us to introduce two notions of Banach modules: AM-modules (or, equivalently, locally convex modules) and AL-modules. In view of the representation theorem of Kakutani for AM- and AL-spaces and the representation result for AM-modules one might expect that, given a measure space X, every AL-module over $\mathrm{L}^\infty(X)$ can be represented as a space $\Gamma^1(E)$ of integrable sections of some measurable Banach bundle E over X. However, this is not the case (see Example 5.10 of [KS19]).
The problem can be solved by taking the lattice theoretic structure into account. As we have seen above, spaces of sections canonically carry a *vector valued norm*, either with values in $\mathrm{U}(K)$ (in the topological case) or in $\mathrm{L}^1(X)$ (in the measurable case). While every AM-module over $\mathrm{C}(K)$ can always be eqipped with such a $\mathrm{U}(K)$-valued norm, AL-modules over $\mathrm{L}^\infty(X)$ generally only admit an $\mathrm{L}^\infty(X)'$-valued norm (see Lemma 3 and Theorem 2 of [Cun67] and Proposition 5.4 of [KS19]). If we restrict ourselves to modules with an $\mathrm{L}^1(X)$-valued norm, then we indeed obtain a representation theorem by employing results of A.E. Gutman (see [Gut93a] and [Gut93b]). We even obtain uniqueness under some separability conditions (see Proposition 5.18 of [KS19]):

Proposition. *For a complete, σ-finite measure space X and an $\mathrm{L}^1(X)$-normed module Γ there is a measurable Banach bundle E over X such that Γ is isometrically isomorphic to $\Gamma^1(E)$. If Γ is separable, then E can be chosen to be separable and is unique up to isometric isomorphy with these properties.*

We now consider the dynamical case. Take a pair $(A;T)$ of a commutative unital C*-algebra A and a unit-preserving *-automorphism $T \in \mathscr{L}(A)$. A *dynamical Banach module* over $(A;T)$ is a pair $(\Gamma;\mathcal{T})$ where Γ is a Banach module over A and $\mathcal{T}$ is a T^{-1}-homomorphism, i.e., $\mathcal{T}(fs) = (T^{-1}f) \cdot \mathcal{T}s$ for all $f \in A$ and $s \in \Gamma$. The results of this and the previous paragraph yield representation theorems for dynamical AM- and AL-modules. In terms of category theory we can formulate them as follows (see Theorem 4.6 and 5.12 of [KS19]).

Theorem. (i) *Given an invertible topological dynamical system $(K;\varphi)$, the assignment*

$$(E;\Phi) \to (\Gamma(E);\mathcal{T}_\Phi)$$

defines a fully faithful and essentially surjective functor from the category of dynamical Banach bundles over $(K;\varphi)$ to the category of dynamical $\mathrm{U}(K)$-normed modules over $(\mathrm{C}(K);T_\varphi)$.

(ii) *Given an invertible measure-preserving system* $(X;\varphi)$ *with* X *complete and separable, the assignemt*

$$(E;\Phi) \to (\Gamma^1(E);\mathcal{T}_\Phi)$$

defines a fully faithful and essentially surjective functor from the catgeory of dynamical Banach bundles over $(X;\varphi)$ *to the category of dynamical* $\mathrm{L}^1(X)$*-normed modules over* $(\mathrm{L}^1(X);T_\varphi)$.

3.5 Structured extensions of dynamical systems

3.5.1 Systems with discrete spectrum

The most structured systems of topological dynamics and ergodic theory are so called systems with discrete spectrum. Recall from the introduction that a power-bounded operator T on a Banach space E has *discrete spectrum* if

$$E = \overline{\mathrm{lin}}\,\{x \in E \mid \text{there is } \lambda \in \mathbb{T} \text{ with } Tx = \lambda x\},$$

i.e., the union of all eigenspaces with respect to unimodular eigenvalues of T are total in E. Topological and measure-preserving dynamical systems have discrete spectrum if the induced Koopman operators on the corresponding spaces $\mathrm{C}(K)$ and $\mathrm{L}^1(X)$, respectively, have discrete spectrum. The group rotations in Standard Example (i) always have discrete spectrum. The following well-known result summarizes several characterizations of topological dynamical systems with discrete spectrum.

Theorem. *For an invertible topological dynamical system* $(K;\varphi)$ *the following assertions are equivalent.*

(a) *The system* $(K;\varphi)$ *is* equicontinuous, *i.e.,* $\{\varphi^n \mid n \in \mathbb{Z}\}$ *is an equicontinuous set of mappings.*

(b) *The system* $(K;\varphi)$ *is* pseudoisometric, *i.e., there is a familiy of invariant pseudometrics generating the topology of* K.

(c) *The Koopman operator* T_φ *has precompact compact orbits* $\{T_\varphi^n f \mid n \in \mathbb{Z}\}$ *for* $f \in \mathrm{C}(K)$.

(d) *The Koopman operator* T_φ *has discrete spectrum.*

3.5.2 Equicontinuous and pseudoisometric extensions

The concepts of equicontinuity and (pseudo)isometry can be relativized to extensions of topological dynamical systems. To do so, we start from an open continuous surjection $q\colon K \to L$ between compact spaces and a uniform space X and define the *space of continuous fiber mappings*

$$\mathrm{C}_q(K,X) := \bigcup_{l\in L} \mathrm{C}(K_l, X),$$

where $K_l := q^{-1}(l) \subseteq K$ is the fiber over $l \in L$. Moreover, if $\vartheta \in \mathrm{C}(K_l, X)$ we set $s(\vartheta) := l \in L$. A subset $\mathcal{F} \subseteq \mathrm{C}_L(K, X)$ is *relatively (uniformly) equicontinuous* if for every entourage U of X there is an entourage V of K such that $(\vartheta(x), \vartheta(y)) \in U$ for every $(x, y) \in V \cap K_{s(\vartheta)} \times K_{s(\vartheta)}$ and each $\vartheta \in \mathcal{F}$. An open extension $q\colon (K; \varphi) \to (L; \psi)$ of invertible topological dynamical systems is then called *equicontinuous* if the set

$$\{\varphi^n|_{K_l} \mid n \in \mathbb{Z}, l \in L\} \subseteq \mathrm{C}_q(K, K)$$

is relatively equicontinuous (cf. Definition 2.9.8 of [Bro79] or Chapter 7 of [Aus88]).
We now generalize the characterization result of the previous section to extensions. As a first step we introduce a "relative compact-open topology" on $\mathrm{C}_q(K, X)$ using the Vietoris topology for the graphs of the mappings in $\mathrm{C}_q(K, X)$ (see Definition 1.15 of [EK19] for the details). This allows us to prove a "relativized" Arzela-Asocoli theorem (see Theorem 1.25 of [EK19]).

Theorem. *Let $q\colon K \to L$ be an open continuous surjection between compact spaces and X a uniform space. For a subset $\mathcal{F} \subseteq \mathrm{C}_q(K, X)$ the following assertions are equivalent.*

(a) *$\mathcal{F}$ is relatively equicontinuous and* $\mathrm{im}(\mathcal{F}) \subseteq X$ *is precompact in X.*

(b) *$\mathcal{F}$ is precompact in* $\mathrm{C}_q(K, X)$.

As a corollary we obtain a characterization of equicontinuous extensions (see Corollary 1.26 of [EK19]).

Corollary. *For an open extension $q\colon (K; \varphi) \to (L; \psi)$ the following statements are equivalent.*

(a) *The extension q is equicontinuous.*

(b) $\{\varphi^n|_{K_l} \mid n \in \mathbb{Z}, l \in L\}$ *is precompact in* $\mathrm{C}_q(K, K)$.

(c) $\{T_\varphi^n f|_{K_l} \mid n \in \mathbb{Z}, l \in L\} \subseteq \mathrm{C}_q(K, \mathbb{C})$ *is relatively equicontinuous for every* $f \in \mathrm{C}(K)$.

(d) $\{T_\varphi^n f|_{K_l} \mid n \in \mathbb{Z}, l \in L\} \subseteq \mathrm{C}_q(K, \mathbb{C})$ *is precompact in* $\mathrm{C}_q(K, \mathbb{C})$ *for every* $f \in \mathrm{C}(K)$.

Besides equicontinuous extensions we can also consider pseudoisometric ones. Here, an extension $q\colon (K; \varphi) \to (L; \psi)$ is *pseudoisometric* if there is a family P of invariant continuous mappings

$$p\colon \{(x, y) \in K \times K \mid q(x) = q(y)\} \to [0, \infty)$$

such that each restriction $p|_{K_l \times K_l}$ for $p \in P$ and $l \in L$ is a pseudometric on K_l and all these restrictions generate the topology of K_l for each $l \in L$ (see Definition 1.3 (c) of [EK19]). For minimal systems the notions of pesudoisometric and equicontinuous extensions coincide (see Corollary 5.10 of [dVr93][6]). However, in general there are equicontinuous extensions which are not pseudoisometric (see Example 1.27 of [EK19]).
Can equicontinuous or pseudoisometric extensions be characterized in terms of some notion of "relative discrete spectrum"? The following theorem due to Knapp (see [Kna67]) gives some hope.

[6] In [dVr93] pseudoisometric extensions are called quasi-isometric.

Theorem. *Let $q\colon (K;\varphi) \to (L;\psi)$ be an equicontinuous extension of minimal, distal and invertible topological dynamical systems. Then the union of all finitely generated closed T_φ-invariant* $\mathrm{C}(L)$*-submodules of* $\mathrm{C}(K)$ *is dense in* $\mathrm{C}(K)$.

What is the connection to the "non-relative" case? If we consider topological systems with discrete spectrum, then $\mathrm{C}(K)$ is the closure of the union of all finite dimensional T_φ-invariant subspaces of $\mathrm{C}(K)$. Replacing the field of complex numbers by the algebra $\mathrm{C}(L)$ and finite dimensional (closed) invariant subspaces over the field $\mathbb{C}$ with finitely generated closed invariant submodules over $\mathrm{C}(L)$, we obtain the notion of *relative discrete spectrum* ocurring in the result above.

Unfortunately, this result is not quite satisfying. Consider the following very simple extension of Standard Example (iii) (see Examples 1.6 6) of [EK19]). We define $K := [0,1] \times \{-1,1\}$ and $L := [0,1]$ as well as $\varphi(x,y) := (x^2, -y)$ for $(x,y) \in K$ and $\psi(x) := x^2$ for $x \in L$. Then the projection $q\colon K \to L$ defines an extension of dynamical systems and is clearly isometric (with respect to the canonical metric on K). Moreover, $\mathrm{C}(K) \cong \mathrm{C}(L)^2$ itself is a finitely generated $\mathrm{C}(L)$-module. The systems $(K;\varphi)$ and $(L;\psi)$, however, are neither minimal nor distal.

We therefore want to weaken the conditions and obtain an operator theretic charaterization of pseudoisometric extensions of more general systems. Looking at the proofs of the above theorems, one observes that the main ingredient is the representation theory of compact groups. As a first step we therefore associate compact groups to extensions of dynamical systems.

3.5.3 Uniform enveloping semigroupoids

As described in Section 2, the Ellis semigroup $\mathrm{E}(K;\varphi)$ of an invertible dynamical system $(K;\varphi)$ contains much information about the system. If the system $(K;\varphi)$ is equicontinuous (e.g., a group rotation as in Standard Example (i)), then $\mathrm{E}(K;\varphi)$ is actually a compact topological group of continuous mappings and agrees with $\mathrm{E}_\mathrm{u}(K;\varphi) := \overline{\{\varphi^n \mid n \in \mathbb{Z}\}} \subseteq \mathrm{C}(K,K)$ where the closure is taken with respect to the topology of uniform convergence (cf. Proposition 2.5 of [Gla07a]). In other words: The *pointwise enveloping semigroup* $\mathrm{E}(K;\varphi)$ agrees with the *uniform enveloping semigroup* $\mathrm{E}_\mathrm{u}(K;\varphi)$ in this case. It turns out that one can carry out similar constructions for extensions if one replaces enveloping semigroups by enveloping *semigroupoids* (see Chapter 2 of [MMMM13] and Definition 1.7 of [EK19]).

Definition. A *semigroupoid* consists of

(i) a set $\mathcal{S}$,

(ii) a set $\mathcal{S}^{(2)} \subseteq \mathcal{S} \times \mathcal{S}$ of *composable pairs*,

(iii) a mapping
$$\cdot\colon \mathcal{S}^{(2)} \to \mathcal{S}, \quad (g,h) \to gh,$$

such that the following asscociativity condition is fulfilled:

If $(g_1,g_2),(g_2,g_3) \in \mathcal{S}^{(2)}$, then $(g_1g_2,g_3),(g_1,g_2g_3) \in \mathcal{S}^{(2)}$ and $(g_1g_2)g_3 = g_1(g_2g_3)$.

If $\mathcal{S}$ is a topological space and $\cdot$ is continuous, then $\mathcal{S}$ is called a *topological semigroupoid*.

Semigroupoids are more general than semigroups[7] in that the multiplication is only partially defined, i.e., we can compute gh only for $(g,h) \in \mathcal{S}^{(2)} \subseteq \mathcal{S} \times \mathcal{S}$.
We now associate semigroupoids with extensions of dynamical systems. Given an open extension $q\colon (K,\varphi) \to (L;\psi)$ of invertible topological dynamical systems, we consider the set

$$\mathrm{C}_q^q(K,K) := \{\vartheta \in \mathrm{C}_q(K,K) \mid \vartheta\colon K_{s(\vartheta)} \to K_{r(\vartheta)} \text{ for some } r(\vartheta) \in L\} \subseteq \mathrm{C}_q(K,K).$$

It canonically becomes a topological semigroupoid with the composition being the composition of mappings whenever it is defined. There is a smallest closed subsemigroupoid $\mathcal{E}_\mathrm{u}(q)$ containing $\{\varphi^n|_{K_l} \mid n \in \mathbb{Z}, l \in L\}$. We call this the *uniform enveloping semigroupoid of* q (see Definition 1.19 in [EK19]).
If q is pseudoisometric, then each element $\vartheta \in \mathcal{E}_\mathrm{u}(q)$ is invertible with $\vartheta^{-1} \in \mathcal{E}_\mathrm{u}(q)$ and the mapping ${}^{-1}\colon \mathcal{E}_\mathrm{u}(q) \to \mathcal{E}_\mathrm{u}(q)$ is a homeomorphism (see Propositions 1.22 and 1.28 of [EK19]). This turns $\mathcal{E}_\mathrm{u}(q)$ into a compact *groupoid* (see Definition 1.1 of [Ren80], Chapter 2 of [MMMM13] or Definition 1.7 of [EK19]).

Definition. A semigroupoid $\mathcal{G}$ together with an *inverse map* ${}^{-1}\colon \mathcal{G} \to \mathcal{G}$ is a *groupoid* if

(i) $(g^{-1}, g) \in \mathcal{G}^{(2)}$ and, if $(g,h) \in \mathcal{G}^{(2)}$, then $g^{-1}(gh) = h$.

(ii) $(g, g^{-1}) \in \mathcal{G}^{(2)}$ and, if $(h,g) \in \mathcal{G}^{(2)}$, then $(hg)g^{-1} = h$.

Moreover, $\mathcal{G}$ is a *topological groupoid* if it is a topological semigroupoid and the mapping ${}^{-1}\colon \mathcal{G} \to \mathcal{G}$ is continuous.

For the example discussed on the previous page we can explicitly compute the uniform enveloping groupoid $\mathcal{E}_\mathrm{u}(q)$. It is given by

$$\mathcal{E}_\mathrm{u}(q) = \{\vartheta_{x_1,x_2} \mid x_1, x_2 \in [0,1]\} \cup \{\varrho_{x_1,x_2} \mid x_1, x_2 \in [0,1]\},$$

where

$$\begin{aligned} \vartheta_{x_1,x_2}&\colon \{x_1\} \times \{-1,1\} \to \{x_2\} \times \{-1,1\}, \quad (x_1,y) \mapsto (x_2,y), \\ \varrho_{x_1,x_2}&\colon \{x_1\} \times \{-1,1\} \to \{x_2\} \times \{-1,1\}, \quad (x_1,y) \mapsto (x_2,-y) \end{aligned}$$

for $x_1, x_2 \in L = [0,1]$.

3.5.4 Representation theory for compact groupoids

In order to characterize pseudoisometric extensions of topological dynamical systems we now focus on the representation theory of compact groupoids which we then apply to the uniform enveloping (semi)groupoids of such extensions. To do so, we introduce the *unit space* of a groupoid $\mathcal{G}$ as the subspace

$$\mathcal{G}^{(0)} := \{g^{-1}g \mid g \in \mathcal{G}\} \subseteq \mathcal{G}$$

[7] Semigroups correspond to the case $\mathcal{S}^{(2)} = \mathcal{S} \times \mathcal{S}$.

as well as the *source* and the *range map*

$$\begin{aligned} s\colon \mathcal{G} &\to \mathcal{G}^{(0)}, \quad g \mapsto g^{-1}g,\\ r\colon \mathcal{G} &\to \mathcal{G}^{(0)}, \quad g \mapsto gg^{-1}. \end{aligned}$$

In the special case of $\mathcal{G} = \mathcal{E}_{\mathrm{u}}(q)$ for an open pseudoisometric extension q of invertible systems we can identify the unit space

$$\mathcal{E}_{\mathrm{u}}(q) = \{\mathrm{id}_{K_l} \mid l \in L\}$$

with L. The range and source maps r and s then just become the mappings

$$r, s\colon \mathcal{E}_{\mathrm{u}}(q) \subseteq \mathrm{C}_q^q(K,K) \to L$$

defined in the previous sections.
One can now introduce the notion of a continuous representation of a compact groupoid $\mathcal{G}$ on a Banach bundle E over its unit space $\mathcal{G}^{(0)}$ as a family of bounded operators $T(g) \in \mathscr{L}(E_{\mathrm{s}(g)}, E_{\mathrm{r}(g)})$ for $g \in \mathcal{G}$ with certain properties (see Definition 3.1 of [Bos11] or Definition 3.4 of [EK19] for the details). For representations of compact groups on Banach spaces the following result holds (see Theorem 15.14 of [EFHN15]).

Theorem. *Let $T\colon G \to \mathscr{L}(E)$ be a strongly continuous representation of a compact group G on a Banach space E. Then the union of all finite-dimensional invariant subspaces of E is dense in E.*

Is a similar result true for representations of compact groupoids on Banach bundles? In general, the representation theory of compact groupoids seems to be quite intricate. It becomes more accessible, however, if we restrict ourselves to compact *transitive* groupoids $\mathcal{G}$, i.e., if we assume that $s^{-1}(u) \cap r^{-1}(v) \neq \emptyset$ for all $u, v \in \mathcal{G}^{(0)}$. In this case we obtain the following result (see Theorem 3.6 of [EK19]).

Theorem. *Let T be a continuous representation of a compact transitive groupoid $\mathcal{G}$ on a Banach bundle E over $\mathcal{G}^{(0)}$. Then the union of all invariant subbundles of constant finite dimension is fiberwise dense in E.*

Returning to extensions of dynamical systems, it turns out that the uniform enveloping groupoid $\mathcal{E}_{\mathrm{u}}(q)$ of an open pseudoisometric extension $q\colon (K;\varphi) \to (L;\psi)$ of invertible systems is transitive if and only if $(L;\psi)$ is topologically ergodic (see Proposition 1.32 of [EK19]). We therefore obtain the following operator theoretic characterization of pseudoisometric extensions (see Theorem 3.8 of [EK19]). Recall that here a module Γ over a unital commutative ring R is *projective* if there is an R-module $\widetilde{\Gamma}$ such that $\Gamma \oplus \widetilde{\Gamma}$ is a free module, i.e., has a basis.

Theorem. *Let $q\colon (K;\varphi) \to (L;\psi)$ be an open extension of invertible topological dynamical systems with $(L;\psi)$ topologically ergodic. Then the following assertions are equivalent.*

(a) *The extension q is pseudoisometric.*

(b) *The union of all finitely generated and projective closed T_φ-invariant* $\mathrm{C}(L)$*-submodules of* $\mathrm{C}(K)$ *is dense in* $\mathrm{C}(K)$.

This result catches the simple example of an isometric extension given above. Moreover, it provides a starting point for a systematic operator theoretic approach to extensions of dynamical systems.

3.6 An outlook: Operator theoretic aspects of extensions

As in topological dynamics, structured extensions also play an important role in ergodic theory and are known as *compact extensions* (see [FKO82], Chapter 6 of [Fur81] or Section 2.13 of [Tao09]), *isometric extensions* (see [Fur77]) or *extensions with relatively discrete spectrum* (see [Zim76]). By the Furstenberg-Zimmer structure theorem every measure-preserving system can then be constructed as a "tower" of such structured extensions and their counterparts, so called *weakly mixing extensions*, see, e.g., Theorem 6.17 of [Fur81] or Theorem 2.15.1 of [Tao09]. This fact can be used to give an ergodic theoretic proof of Szemerédi's theorem discussed in the introduction (see [FKO82]).

As a continuation of our joint project, Nikolai Edeko and myself will now focus on this structure theory for measure-preserving systems and—using our methods—develop a systematic operator theoretic approach to extensions of dynamical systems.

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Appendix

1 Accepted Publications

1.1 Compact operator semigroups applied to dynamical systems

COMPACT OPERATOR SEMIGROUPS APPLIED TO DYNAMICAL SYSTEMS

HENRIK KREIDLER

Abstract. In this paper we develop a systematic theory of compact operator semigroups on locally convex vector spaces. In particular we prove new and generalized versions of the mean ergodic theorem and apply them to different notions of mean ergodicity appearing in topological dynamics.

Mathematics Subject Classification (2010). Primary 47A35, 47D03; Secondary 37B05.

1. Introduction

In [Kö94] and [Kö95] A. Köhler introduced the *enveloping operator semigroup* of a power bounded operator $T \in \mathscr{L}(X)$ on a Banach space X as the closure of the semigroup $\mathcal{S} := \{(T')^n \mid n \in \mathbb{N}_0\}$ with respect to the operator topology of pointwise convergence induced by the weak* topology on X'.
If the operator T is the Koopman operator of a topological dynamical system $(K;\varphi)$, i.e., $Tf = f \circ \varphi$ for each $f \in \mathrm{C}(K)$, the relation between this enveloping operator semigroup and the classical *Ellis semigroup* is an interesting issue (see [Kö95] and [Gla07a]).
In addition, A. Romanov used the semigroup given by the convex closure of $\mathcal{S}$ to examine mean ergodicity of operators with respect to the weak* topology (see [Rom11]) and—in the setting of topological dynamics—with respect to the topology of pointwise convergence (see [Rom16]).

In this paper we generalize the concepts from above and give a new and systematic approach to compact operator semigroups on locally convex spaces. We then discuss its applications to dynamical systems.
It turns out that these semigroups are right topological semigroups (see [BJM78] and [BJM89] for an introduction to their theory) with respect to the topology of pointwise convergence. We use this as our starting point and introduce the new concept of abstract Köhler semigroups with their basic properties (see Proposition 2.4 and Proposition 2.5). In the subsequent section we discuss examples appearing in topological dynamics: the *Köhler semigroup*, the *Ellis semigroup* and the *Jacobs semigroup* (see Proposition 3.8). We give a new operator theoretic characterization of the Ellis semigroup (see Definition 3.4) and prove that a metric topological dynamical system is tame if and only its convex Köhler semigroup is a Fréchet–Urysohn space (see Proposition 3.11).

The author wants to thank Roland Derndinger, Nikolai Edeko, Rainer Nagel and Alexander Romanov for ideas, helpful suggestions and inspiring discussions. The author is also grateful to the referee for their suggestions.

Our main results are contained in the last two sections. In Section 4 we return to the abstract viewpoint and generalize a mean ergodic theorem established by A. Romanov in [Rom11] for a single operator on a dual Banach space with the weak* topology to left amenable operator semigroups acting on locally convex spaces (see Theorem 4.3). The result is then applied to extend a mean ergodic theorem of M. Schreiber ([Sch13b]) to right amenable operator semigroups on barrelled locally convex spaces (see Theorem 4.7).
In the final section we study unique ergodicity, norm mean ergodicity and weak* mean ergodicity for right amenable semitopological semigroups acting on a compact space. This approach as well as the result that transitive weak* mean ergodic systems are already uniquely ergodic (see Theorem 5.6) seem to be new.
At the end of this section we discuss mean ergodicity of tame metric systems. We give a new proof for unique ergodicity of minimal tame metric dynamical systems generalizing results of Glasner ([Gla07a]), Huang ([Hua06]) as well as Kerr and Li ([KL07]) to the case of amenable semigroup actions (see Corollary 5.4). Finally we extend results of Romanov in [Rom16] and characterize weak* mean ergodicity for tame amenable semigroup actions (see Theorem 5.10).

In this paper all vector spaces are complex. Moreover, all topological vector spaces and compact spaces are assumed to be Hausdorff. Given topological vector spaces X and Y we denote the continuous linear mappings from X to Y by $\mathscr{L}(X,Y)$ and set $\mathscr{L}(X) \coloneqq \mathscr{L}(X,X)$.
We recall that a special class of locally convex spaces arises from dual pairs (see [Sch99], Chapter IV). More generally, take two vector spaces X and Y and a bilinear mapping $\langle \cdot\,,\cdot \rangle \colon X \times Y \longrightarrow \mathbb{C}$ that separates X, i.e., for each $0 \neq x \in X$ there is $y \in Y$ with $\langle x, y\rangle \neq 0$, and call the pair (X,Y) a *left-separating pair.* The seminorms ρ_y given by $\rho_y(x) \coloneqq |\langle x, y\rangle|$ for $x \in X$ and $y \in Y$ define a (Hausdorff) locally convex topology $\sigma(X,Y)$ on X.
The canonical duality between a Banach space X and its dual X' clearly defines left-separating pairs (X,X') and (X',X). Other examples are discussed in Section 3.

2. Abstract Köhler Semigroups

We start with a general definition of Köhler semigroups. Examples and applications are given in the sections below.

Definition 2.1. Let X be a locally convex space, $\mathcal{S} \subseteq \mathscr{L}(X)$ a subsemigroup and equip X^X with the product topology. The *(abstract) Köhler semigroup* of $\mathcal{S}$ is the closure $\mathcal{K}(\mathcal{S}) \coloneqq \overline{\mathcal{S}} \subseteq X^X$.

If X carries the weak topology induced by a left-separating pair (X,Y) we denote the Köhler semigroup associated to a semigroup $\mathcal{S} \subseteq \mathscr{L}(X)$ by $\mathcal{K}(\mathcal{S}; X, Y)$.

Recall that a semigroup S equipped with a topology is called *right topological* if the mapping

$$S \longrightarrow S, \quad s \mapsto st$$

is continuous for each $t \in S$. It is *left topological* if

$$S \longrightarrow S, \quad s \mapsto ts$$

is continuous for each $t \in S$ and *semitopological* if it is both, left topological and right topological (see Section 1.3 of [BJM89]).
The *topological center* $\Lambda(S)$ of a right topological semigroup S is

$$\Lambda(S) := \{s \in S \mid t \mapsto st \text{ is continuous}\}.$$

The following basic properties of the Köhler semigroup are readily verified.

Lemma 2.2. *For an operator semigroup $\mathcal{S} \subseteq \mathscr{L}(X)$ on a locally convex space X the following assertions hold.*

(i) *$\mathcal{K}(\mathcal{S})$ is a right topological semigroup.*

(ii) *The topological center $\Lambda(\mathcal{K}(\mathcal{S}))$ contains $\mathcal{S}$.*

(iii) *$\mathcal{K}(\mathcal{S})$ is compact if and only if $\mathcal{S}x$ is relatively compact for each $x \in X$.*

(iv) *If $\mathcal{K}(\mathcal{S}) \subseteq \mathscr{L}(X)$, then $\mathcal{K}(\mathcal{S})$ is semitopological.*

(v) *If $\mathcal{S}$ is abelian, then $\mathcal{K}(\mathcal{S})$ is semitopological if and only if it is abelian.*

For operator semigroups on certain classes of locally convex spaces one can say more and we recall some definitions from this theory. For an introduction we refer to [Sch99] and [Jar81].

Definition 2.3. A locally convex vector space X is called

(i) *barrelled* if every radial, convex, circled and closed set is a zero neighborhood.

(ii) *Montel* if it is barrelled and every bounded subset is relatively compact.

(iii) *quasi-complete* if every closed bounded subset is complete (with respect to the uniformity defined by the zero neighborhoods of X).

Banach spaces and even Fréchet spaces are barrelled. Important examples of Montel spaces are the space of all holomorphic functions $\mathrm{H}(\Omega)$ on an open connected subset $\Omega \subseteq \mathbb{C}$ equipped with the topology of compact convergence and the space of smooth functions $\mathrm{C}^\infty(\Omega)$ on an open set $\Omega \subseteq \mathbb{R}^n$ equipped with the topology of compact convergence and, in all derivatives. We refer to Section 11.5 of [Jar81] for more about these spaces.
Besides Banach and Fréchet spaces the dual space of a Banach space equipped with the weak* topology is quasi-complete.

Proposition 2.4. *Consider an operator semigroup $\mathcal{S} \subseteq \mathscr{L}(X)$ on a locally convex space X.*

(i) *The Köhler semigroup $\mathcal{K}(\mathcal{S})$ consists of continuous mappings on X if one of the following conditions is fulfilled.*

  (a) *For each pointwise bounded subset $M \subseteq \mathscr{L}(X)$ the pointwise closure $\overline{M} \subseteq X^X$ is contained in $\mathscr{L}(X)$.*

  (b) *The space X is barrelled.*

  (c) *X carries the weak topology $\sigma(X, X')$ associated to a barrelled topology on X.*

(d) *The semigroup* $\mathcal{S} \subseteq \mathscr{L}(X)$ *is equicontinuous.*

(ii) *If* X *is Montel, then* $\mathcal{K}(\mathcal{S})$ *is a compact semitopological subsemigroup of* $\mathscr{L}(X)$.

Proof. Clearly, condition (a) implies $\mathcal{K}(\mathcal{S}) \subseteq \mathscr{L}(X)$. We show that the spaces in (b) and (c) satisfy (a). For a barrelled space X this is a consequence of the Banach–Steinhaus Theorem (see Theorem III.4.6 in [Sch99]).
Now consider a pointwise bounded set M of $\sigma(X, X')$-continuous operators and take $T_\alpha \in M$ for $\alpha \in A$ and $T \in X^X$ with $\lim_\alpha \langle T_\alpha x, x' \rangle = \langle Tx, x' \rangle$ for all $x \in X$ and $x' \in X'$. By passing to the (algebraic) adjoint operator $T^*\colon X' \longrightarrow X^*$, we obtain

$$T^* x'(x) = \lim_\alpha T'_\alpha x'(x)$$

for $x' \in X'$ and $x \in X$.
For given $x' \in X'$ the net $(T'_\alpha x')_{\alpha \in A}$ is contained in the bounded set $M'x' = \{S'x' \mid S \in M\} \subseteq X'$ which is (since X is barrelled) relatively $\sigma(X', X)$-compact (see the corollary to IV.1.6 in [Sch99]). Thus we find a subnet of $(T_\alpha x')_{\alpha \in A}$ converging in the $\sigma(X', X)$-topology to an element $y' \in X'$ which yields $T^* x' = y' \in X'$. We obtain $T^* X' \subseteq X'$ which shows that T is $\sigma(X, X')$-continuous (IV.2.1 in [Sch99]) and so (ii) is proved.
In the situation of (d) we obtain $\mathcal{K}(\mathcal{S}) \subseteq \mathscr{L}(X)$ by Theorem III.4.3 of [Sch99]. From the definition of a Montel space and part (i) we immediately obtain (ii). □

At the end of this section we prove some relations between different Köhler semigroups. We recall that a continuous mapping between two right topological semigroups is a *homomorphism of right topological semigroups* if it is multiplicative. A surjective homomorphism is called *epimorphism.*

Proposition 2.5. *Let* $\mathcal{S}_i \subseteq \mathscr{L}(X_i)$ *be operator semigroups on locally convex spaces* X_i *for* $i = 1, 2$.

(i) *Assume that* X_2 *is a subspace of* X_1 *and* $\mathcal{S}_2 = \mathcal{S}_1|_{X_2}$. *If* $\mathcal{K}(\mathcal{S}_1)$ *and* $\mathcal{K}(\mathcal{S}_2)$ *are compact, then*

$$\mathcal{K}(\mathcal{S}_1) \longrightarrow \mathcal{K}(\mathcal{S}_2), \quad S \mapsto S|_{X_2}$$

is an epimorphism of right topological semigroups.

(ii) *Assume that* $\Phi \in \mathscr{L}(X_1, X_2)$ *is a surjective map and* $\Psi\colon \mathcal{S}_1 \longrightarrow \mathcal{S}_2$ *is an epimorphism of semigroups with* $\Phi(Sx) = \Psi(S)\Phi(x)$ *for all* $S \in \mathcal{S}_1$ *and* $x \in X_1$. *If* $\mathcal{K}(\mathcal{S}_1)$ *is compact, then*

$$\hat{\Psi}\colon \mathcal{K}(\mathcal{S}_1) \longrightarrow \mathcal{K}(\mathcal{S}_2), \quad S \mapsto \hat{\Psi}(S)$$

with $\hat{\Psi}(S)(\Phi(x)) := \Phi(Sx)$ *for* $S \in \mathcal{S}_1$ *and* $x \in X$ *is an epimorphism of right topological semigroups with* $\hat{\Psi}|_{\mathcal{S}_1} = \Psi$.

Proof. We start with (i). The mapping given by

$$\pi\colon \mathcal{K}(\mathcal{S}_1) \longrightarrow X_1^{X_2}, \quad S \mapsto S|_{X_2}$$

is clearly continuous and by compactness of $\mathcal{K}(\mathcal{S}_1)$ closed. Compactness of $\mathcal{K}(\mathcal{S}_2)$ implies

$$\mathcal{K}(\mathcal{S}_2) = \overline{\mathcal{S}_2}^{X_2^{X_2}} = \overline{\mathcal{S}_2}^{X_1^{X_2}}$$

and thus

$$\pi(\mathcal{K}(\mathcal{S}_1)) = \pi(\overline{\mathcal{S}_1}) = \overline{\pi(\mathcal{S}_1)}^{X_1^{X_2}} = \overline{\mathcal{S}_2}^{X_2^{X_2}} = \mathcal{K}(\mathcal{S}_2).$$

Multiplicativity of the restriction map is trivial and (i) follows.
We now prove (ii). To this end, we first show that

$$\hat{\Psi} \colon \mathcal{K}(\mathcal{S}_1) \longrightarrow X_2^{X_2}, \quad S \mapsto \hat{\Psi}(S)$$

is well-defined. Take $x, y \in X_1$ with $\Phi(x) = \Phi(y)$ and $S \in \mathcal{K}(\mathcal{S}_1)$ and take $S_\alpha \in \mathcal{S}_1$ for $\alpha \in A$ such that $\lim_\alpha S_\alpha z = Sz$ for each $z \in X_1$. We then obtain

$$\begin{aligned}\Phi(Sx) &= \Phi\left(\lim_\alpha S_\alpha x\right) = \lim_\alpha \Phi(S_\alpha x) = \lim_\alpha \Psi(S_\alpha)\Phi(x) \\ &= \lim_\alpha \Psi(S_\alpha)\Phi(y) = \Phi\left(\lim_\alpha S_\alpha y\right) = \Phi(Sy).\end{aligned}$$

This proves that $\hat{\Psi}$ is well-defined. Continuity of $\hat{\Psi}$ is obvious and by the same arguments as in (i) we obtain $\hat{\Psi}(\mathcal{K}(\mathcal{S}_1)) = \mathcal{K}(\mathcal{S}_2)$. To finish the proof, consider $S, T \in \mathcal{K}(\mathcal{S}_1)$. We then have

$$\hat{\Psi}(ST)(\Phi(x)) = \Phi(STx) = \hat{\Psi}(S)(\Phi(Tx)) = \hat{\Psi}(S)(\hat{\Psi}(T)(\Phi(x))$$

for each $x \in X$ and thus $\hat{\Psi}(ST) = \hat{\Psi}(S)\hat{\Psi}(T)$. □

3. Compact Operator Semigroups in Topological Dynamics

We now apply the concepts and results from Section 2 in topological dynamics.

Definition 3.1. A *topological dynamical system* is a pair $(K; S)$ of a compact space K and a semitopological semigroup S acting on K such that the mapping

$$S \times K \longrightarrow K, \quad (s, x) \mapsto sx$$

is continuous.

Occasionally we identify an element $s \in S$ with the continuous mapping $K \longrightarrow K$, $x \mapsto sx$.

Definition 3.2. Let $(K; S)$ be a topological dynamical system.

(i) To each $s \in S$ we associate the *Koopman operator* $T_s \in \mathscr{L}(\mathrm{C}(K))$ by setting $T_s f(x) \coloneqq f(sx)$ for $s \in S$ and $x \in K$.

(ii) The *Koopman semigroup* $\mathcal{T}_S$ associated to $(K; S)$ is

$$\mathcal{T}_S \coloneqq \{T_s \mid s \in S\} \subseteq \mathscr{L}(\mathrm{C}(K)).$$

For an introduction to the theory of dynamical systems we refer to [Ell69], [Bro70] and [Gla08]. For an operator theoretic approach to this topic see [EFHN15].
We now consider the adjoint semigroup $\mathcal{T}_S' \coloneqq \{T_s' \mid s \in S\} \subseteq \mathscr{L}(\mathrm{C}(K)')$

on the dual space $\mathrm{C}(K)'$ which we identify with the Banach lattice of regular Borel measures on K. We also identify the Banach lattice of discrete measures on K with

$$\ell^1(K) := \left\{ (a_x)_{x\in K} \in \mathbb{C}^K \,\middle|\, \sum_{x\in K} |a_x| < \infty \right\}.$$

Given an S-invariant probability measure $\mu \in \mathrm{C}(K)'$ we identify the Banach lattice of measures absolutely continuous with respect to μ with $\mathrm{L}^1(K,\mu)$. We then obtain operator semigroups associated to the left-separating pairs $(\mathrm{C}(K)', \mathrm{C}(K))$, $(\ell^1(K), \mathrm{C}(K))$ and $(\mathrm{L}^1(K,\mu), \mathrm{C}(K))$.

Lemma 3.3. *Consider a topological dynamical system $(K;S)$ and the corresponding Koopman semigroup $\mathcal{T}_S$. The following assertions are true.*

(i) *The semigroup $\mathcal{T}_S'$ has relatively $\sigma(\mathrm{C}(K)', \mathrm{C}(K))$-compact convex orbits $\mathrm{co}\,\mathcal{T}_S'\mu$ for all $\mu \in \mathrm{C}(K)'$.*

(ii) *The space $\ell^1(K)$ is $\mathcal{T}_S'$-invariant and the semigroup $\mathcal{T}_S'|_{\ell^1(K)}$ has relatively $\sigma(\ell^1(K), \mathrm{C}(K))$-compact orbits $\mathcal{T}_S'(a_x)_{x\in K}$ for all $(a_x)_{x\in K} \in \ell^1(K)$.*

(iii) *For each S-invariant probability measure $\mu \in \mathrm{C}(K)'$ the space $\mathrm{L}^1(K,\mu)$ is $\mathcal{T}_S'$-invariant and the semigroup $\mathcal{T}_S'|_{\mathrm{L}^1(K,\mu)}$ has relatively $\sigma(\mathrm{L}^1(K,\mu), \mathrm{C}(K))$-compact convex orbits $\mathrm{co}\,\mathcal{T}_S'h$ for all $h \in \mathrm{L}^1(K,\mu)$.*

Proof. The first assertion holds since all operators in $\mathcal{T}_{\mathcal{S}}$ are contractions and the unit ball of $\mathrm{C}(K)'$ is weak* compact by the Banach–Alaoglu theorem (see Theorem III.4.3 of [Sch99]).

Since $T_s'\delta_x = \delta_{sx}$ for $s \in S$ and $x \in K$ we obtain that $\ell^1(K)$ is $\mathcal{T}_S'$-invariant. Now take $(a_x)_{x\in K} \in \ell^1(K)$ and a net $(T_{s_\alpha})_{\alpha\in A}$. Since K^K is compact for the product topology, we find a subnet $(s_\beta)_{\beta\in B}$ of $(s_\alpha)_{\alpha\in A}$ such that $(s_\beta x)_{\beta\in B}$ converges to some element $s(x) \in K$ for each $x \in K$. Now consider $(b_x)_{x\in K} \in \ell^1(K)$ given by

$$b_x = \sum_{\substack{y\in K\\ s(y)=x}} a_y$$

for $x \in K$. Let $\varepsilon > 0$, $f \in \mathrm{C}(K)$ and choose $x_1, ..., x_N \in K$ with

$$\sum_{x\notin\{x_1,\dots,x_N\}} |a_x| \cdot 2\|f\| \leqslant \varepsilon.$$

We find $\beta_0 \in B$ with $|f(s_\beta x_j) - f(s(x_j))| \cdot \sum_{k=1}^N |a_{x_k}| \leqslant \varepsilon$ for all $\beta \geqslant \beta_0$ and $j \in \{1, ..., N\}$. We thus obtain

$$\left| \langle f, T_{s_\beta}'(a_x)_{x\in K} - (b_x)_{x\in K}\rangle \right| \leqslant \sum_{x\in K} |a_x| \cdot |f(s_\beta x) - f(s(x))| \leqslant 2\varepsilon$$

for all $\beta \geqslant \beta_0$. This shows $\lim_\beta T_{s_\beta}'(a_x)_{x\in K} = (b_x)_{x\in K}$.

Finally consider an S-invariant probability measure $\mu \in \mathrm{C}(K)'$. If a measure $\nu \in \mathrm{C}(K)'$ is absolutely continuous with respect to μ, then so is $T_s'\nu$ for each $s \in S$. To see this, take a Borel measurable μ-null set A. Since μ is invariant, we obtain $\mu(s^{-1}(A)) = 0$ and by absolute continuity of ν we

conclude $T_s'\nu(A) = \nu(s^{-1}(A)) = 0$.
To check compactness we note that the set

$$\mathrm{co}\,\mathcal{T}_S'|_{\mathrm{L}^1(\mathrm{K},\mu)} = \mathrm{co}\,\{T_s'|_{\mathrm{L}^1(K,\mu)} \mid s \in S\}$$

consists of bi-Markov operators on $\mathrm{L}^1(K,\mu)$ and thus is relatively compact with respect to the weak operator topology on $\mathrm{L}^1(K,\mu)$ (see Theorem 13.8 in [EFHN15]). Therefore $\mathrm{co}\,\mathcal{T}_S'h$ is relatively $\sigma(\mathrm{L}^1(K,\mu),\mathrm{L}^\infty(K,\mu))$-compact and in particular relatively $\sigma(\mathrm{L}^1(K,\mu),\mathrm{C}(K))$-compact for each $h \in \mathrm{L}^1(K,\mu)$. □

We now introduce various enveloping semigroups.

Definition 3.4. To a topological dynamical system $(K;S)$ we associate the following semigroups.

(i) The *Köhler semigroup* $\mathcal{K}(K;S) := \mathcal{K}(\mathcal{T}_S';\mathrm{C}(K)',\mathrm{C}(K))$.

(ii) The *convex Köhler semigroup* $\mathcal{K}_{\mathrm{c}}(K;S) := \mathcal{K}(\mathrm{co}\,\mathcal{T}_S';\mathrm{C}(K)',\mathrm{C}(K))$.

(iii) The *Ellis semigroup* $\mathcal{E}(K;S) := \mathcal{K}(\mathcal{T}_S';\ell^1(K),\mathrm{C}(K))$.

(iv) For an S-invariant probability measure $\mu \in \mathrm{C}(K)'$ the *Jacobs semigroup* $\mathcal{J}(K,\mu;S) := \mathcal{K}(\mathcal{T}_S';\mathrm{L}^1(K,\mu),\mathrm{C}(K))$.

(v) For an S-invariant probability measure $\mu \in \mathrm{C}(K)'$ the *convex Jacobs semigroup* $\mathcal{J}_{\mathrm{c}}(K,\mu;S) := \mathcal{K}(\mathrm{co}\,\mathcal{T}_S';\mathrm{L}^1(K,\mu),\mathrm{C}(K))$.

Remark 3.5. In the proof of Lemma 3.3 we have already seen that $\mathcal{T}_S'$ consists of bi-Markov operators and this set is relatively compact with respect to the weak operator topology (see Theorem 13.8 in [EFHN15]). Since the $\sigma(\mathrm{L}^1(K,\mu),\mathrm{C}(K))$-topology is coarser than the $\sigma(\mathrm{L}^1(K,\mu),\mathrm{L}^\infty(K,\mu))$-topology, we immediately obtain

$$\mathcal{J}(K,\mu;S) = \mathcal{K}(\mathcal{T}_S';\mathrm{L}^1(K,\mu),\mathrm{L}^\infty(K,\mu)) \subseteq \mathscr{L}(\mathrm{L}^1(K,\mu)),$$
$$\mathcal{J}_{\mathrm{c}}(K,\mu;S) = \mathcal{K}(\mathrm{co}\,\mathcal{T}_S';\mathrm{L}^1(K,\mu),\mathrm{L}^\infty(K,\mu)) \subseteq \mathscr{L}(\mathrm{L}^1(K,\mu)).$$

In particular, the Jacobs and the convex Jacobs semigroups are semitopological.

Remark 3.6. Consider the classical Ellis semigroup $\mathrm{E}(K;S)$ of a topological dynamical system $(K;S)$ given as the closure $\overline{S} \subseteq K^K$ (see [Ell60]). If we identify K with its homeomorphic copy $\{\delta_x \mid x \in K\} \subseteq \mathrm{C}(K)'$ (where $\mathrm{C}(K)'$ is equipped with the weak* topology), one readily checks that the mapping

$$\mathcal{E}(K;S) \longrightarrow \mathrm{E}(K;S), \quad R \mapsto R|_K$$

is an isomorphism of right topological semigroups (this is a simple consequence of the fact, that the linear hull of Dirac measures is norm dense in $\ell^1(K)$). Thus our Ellis semigroup is isomorphic to the classical one, but is now obtained as an operator semigroup.

Remark 3.7. Proposition 2.5 shows that taking subsystems and factors of a topological dynamical system $(K;S)$ induces epimorphisms of the corresponding Köhler and Ellis semigroups.

The following result is an immediate consequence of Proposition 2.5 (i) and shows the relations between these semigroups.

Proposition 3.8. *Let $(K;S)$ be a topological dynamical system and $\mu \in \mathrm{C}(K)'$ an S-invariant probability measure. We then have the following diagram of epimorphisms of right topological semigroups where the arrows are given by the canonical restriction maps.*

$$\begin{tikzcd}
 & \mathcal{K}(K;S) \arrow[dl] \arrow[dr] & \subseteq & \mathcal{K}_{\mathrm{c}}(K;S) \arrow[dr] & \\
\mathcal{E}(K;S) & & \mathcal{J}(\mu) & \subseteq & \mathcal{J}_{\mathrm{c}}(\mu)
\end{tikzcd}$$

It is natural to ask when these epimorphisms are isomorphisms. If the topological dynamical system $(K;S)$ is *weakly almost periodic*, i.e., if $\mathcal{T}_S f$ is relatively weakly compact for each $f \in \mathrm{C}(K)$, we obtain the following.

Proposition 3.9. *If $(K;S)$ is a weakly almost periodic system, then the canonical epimorphism $\mathcal{K}(K;S) \longrightarrow \mathcal{E}(K;S)$ is an isomorphism.*
If moreover $\mu \in \mathrm{C}(K)'$ is a strictly positive invariant probability measure, then the canonical epimorphisms $\mathcal{K}(K;S) \longrightarrow \mathcal{J}(\mu)$ and $\mathcal{K}_{\mathrm{c}}(K;S) \longrightarrow \mathcal{J}_{\mathrm{c}}(\mu)$ are also isomorphisms.

Proof. Since $(K;S)$ is weakly almost periodic, $\mathcal{K}(\mathcal{T}_S;\mathrm{C}(K),\mathrm{C}(K)')$ is a compact semitopological semigroup and one readily checks that

$$\mathcal{K}(\mathcal{T}_S;\mathrm{C}(K),\mathrm{C}(K)') \longrightarrow \mathcal{K}(\mathcal{T}_S';\mathrm{C}(K)',\mathrm{C}(K)) = \mathcal{K}(K;S), \quad S \mapsto S'$$

is an isomorphism if the order of multiplication in $\mathcal{K}(\mathcal{T}_S;\mathrm{C}(K),\mathrm{C}(K)')$ is reversed. Thus each operator in $\mathcal{K}(K;S)$ is weak* continuous. Since $\ell^1(K)$ is weak* dense $\mathrm{C}(K)'$, we obtain injectivity of the restriction map $\mathcal{K}(K;S) \longrightarrow \mathcal{E}(K;S)$.
If $\mu \in \mathrm{C}(K)'$ is a strictly positive invariant probability measure, then $\mathrm{L}^1(K,\mu)$ is also weak* dense in $\mathrm{C}(K)'$, and thus $\mathcal{K}(K;S) \longrightarrow \mathcal{J}(\mu)$ is injective.
Since, by Krein's theorem (see Theorem 11.4 in [Sch99] or Theorem C.11 in [EFHN15]), the sets $\mathrm{co}\,\mathcal{T}_S f$ are also relatively weakly compact, a similar argument shows that the remaining map is an isomorphism. □

The question when the epimorphism $\mathcal{K}(K;S) \longrightarrow \mathcal{E}(K;S)$ is an isomorphism was first posed by J. S. Pym ([Pym89]) and then answered by A. Köhler ([Kö95]) for metrizable topological dynamical systems satisfying the following condition which is weaker than weak almost periodicity.

Definition 3.10. A metric topological dynamical system $(K;S)$ is *tame* if for each $f \in \mathrm{C}(K)$ the orbit $\mathcal{T}_S f$ is relatively sequentially compact with respect to the product topology of $\mathbb{C}^K$.

Such systems have been studied in detail by E. Glasner and M. Megrelishvili ([GM06], [Gla06], [Gla07a], [Gla07b], [GM12], [GM13] and [GM15]) and later by W. Huang in [Hua06], D. Kerr and H. Li in [KL07] as well as by A. Romanov in [Rom16].
We recall that a topological space X is a *Fréchet–Urysohn space* (see page 53 of [Eng89]) if each subset $A \subseteq X$ satisfies

$$\overline{A} = \left\{ x \in X \mid \text{ there is a sequence } (x_n)_{n\in\mathbb{N}} \text{ in } A \text{ with } x = \lim_{n\to\infty} x_n \right\}.$$

A compact space K is called *Rosenthal compact* if it can be continuously embedded into the space of Baire 1 functions $\mathrm{B}_1(X)$ on a Polish space X. By results of J. Bourgain, D. H. Fremlin and M. Talagrand (see [BFT78]) every Rosenthal compact space is a Fréchet–Urysohn space. Moreover, closed subspaces and countable products of Rosenthal compact spaces are Rosenthal compact (see Section c-17 of [HNV03] for more properties of such spaces). This leads to the following characterizations of tameness.

Proposition 3.11. *For a metric topological dynamical system* $(K;S)$ *the following are equivalent.*

(i) *The system* $(K;S)$ *is tame.*

(ii) $\mathcal{K}_{\mathrm{c}}(K;S)$ *is a Rosenthal compact space.*

(iii) $\mathcal{K}(K;S)$ *is a Rosenthal compact space.*

(iv) $\mathcal{E}(K;S)$ *is a Fréchet–Urysohn space.*

Proof. As above we identify K with its homeomorphic copy $\{\delta_x \mid x \in K\}$ in $\mathrm{C}(K)'$. In particular, $(R'f|_K)(x) = \langle f, R\delta_x\rangle$ for $R \in \mathcal{K}_{\mathrm{c}}(K;S)$, $f \in \mathrm{C}(K)$ and $x \in K$.

Assume that (i) holds. Then, for each $f \in \mathrm{C}(K)$ the set $\mathrm{co}\,\mathcal{T}_S f$ is also relatively sequentially compact with respect to the product topology of $\mathbb{C}^X$ and its closure is contained in the space of Baire 1 functions $\mathrm{B}_1(K)$ (see Corollary 5G of [BFT78]). Take a dense subset $\{f_n \mid n \in \mathbb{N}\} \subseteq \mathrm{C}(K)$ (which is possible, since K is metric). We will show that

$$\Phi\colon \mathcal{K}_{\mathrm{c}}(K;S) \longrightarrow \prod_{k\in\mathbb{N}} \overline{\mathrm{co}\,\mathcal{T}_S f_k}^{\mathbb{C}^K}, \quad R \mapsto (R'f_k|_K)_{k\in\mathbb{N}}$$

is a continuous embedding. Then, since countable products and closed subspaces of Rosenthal compact spaces are Rosenthal compact, this will finally prove that $\mathcal{K}_{\mathrm{c}}(K;S)$ is Rosenthal compact.

We first check that Φ is well-defined. To this end, take a net $(R_\alpha')_{\alpha\in A} \in \mathrm{co}\,\mathcal{T}_S'$ converging to $R \in \mathcal{K}_{\mathrm{c}}(K;S)$. We then obtain

$$(R'f|_K)(x) = \langle R'f, \delta_x\rangle = \lim_\alpha \langle R_\alpha f, \delta_x\rangle = \lim_\alpha (R_\alpha f)(x)$$

for all $f \in \mathrm{C}(K)$ and $x \in K$. Thus Φ is well-defined and it is clearly continuous. Given a net $(R_\alpha')_{\alpha\in A} \in \mathrm{co}\,\mathcal{T}_S'$ converging to $R \in \mathcal{K}_{\mathrm{c}}(K;S)$ the main theorem of [Ros77] implies

$$\int_K (R'f_k)|_K \,\mathrm{d}\mu = \lim_\alpha \int_K R_\alpha f_k \,\mathrm{d}\mu = \lim_\alpha \langle R_\alpha f_k, \mu\rangle = \langle f_k, R\mu\rangle$$

for each $k \in \mathbb{N}$ and $\mu \in \mathrm{C}(K)'$. This shows that Φ is injective.

Since closed subspaces of Rosenthal compact spaces are Rosenthal compact, we immediately obtain that (ii) implies (iii). Moreover, continuous images of compact Fréchet–Urysohn spaces are again Fréchet–Urysohn, so (iii) implies (iv) by Proposition 3.8.

Finally assume $\mathcal{E}(K;S)$ to be Fréchet–Urysohn. Take $f \in \mathrm{C}(K)$ and a net $(T_{s_\alpha} f)_{\alpha\in A}$ with $s_\alpha \in S$ for each $\alpha \in A$ converging pointwise to $g \in \mathbb{C}^K$. By passing to a subnet we may assume that $(s_\alpha)_{\alpha\in A}$ converges pointwise to some $\psi \in \mathrm{E}(K;S)$ such that $g = f \circ \psi$. Since $\mathrm{E}(K;S)$ is a Fréchet–Urysohn space, we find a sequence $(s_n)_{n\in\mathbb{N}}$ in S converging to ψ whereby

$$g = f \circ \psi = \lim_{n\to\infty} f \circ s_n \in \mathrm{B}_1(K).$$

Thus $\overline{\mathcal{T}_S f_k}^{\mathbb{C}^K} \subseteq \mathrm{B}_1(K)$ and Corollary 5G of [BFT78] yields the claim. □

Remark 3.12. The equivalence of assertions (i) and (iv) is well-known. Our proof is based on the arguments of Glasner and Megrelishvili (see Theorem 3.2 in [GM06]), but extends the result to the convex Köhler semigroup showing thereby that the classes D2 and D3 of dynamical systems in [Rom16] are actually the same.

The next result is known (see Theorem 1.5 of [Gla06] for the case of a group action), but for the sake of completeness we give a short proof.

Proposition 3.13. *Let $(K;S)$ be a tame metric topological dynamical system and for $\psi \in \mathrm{E}(K;S)$ define the* Koopman operator $T_\psi \in \mathscr{L}(\mathrm{C}(K), \mathrm{B}_1(K))$ *by $T_\psi f := f \circ \psi$ for $\psi \in \mathrm{E}(K;S)$ and $f \in \mathrm{C}(K)$. Then*

$$J\colon \mathrm{E}(K;S) \longrightarrow \mathcal{K}(K;S), \quad \psi \mapsto T_\psi'|_{\mathrm{C}(K)'}$$

is an epimorphism of right topological semigroups and is the inverse to the canonical restriction map

$$\mathcal{K}(K;S) \longrightarrow \mathrm{E}(K;S), \quad S \mapsto S|_K.$$

Proof. A map from a sequential space to a Hausdorff space is continuous if and only if it is sequentially continuous (see Proposition 1.6.15 in [Eng89]). It is therefore a direct consequence of Lebesgue's Theorem that J is continuous. Multiplicativity is trivial and, since $J(S) = \mathcal{T}_S'$, J is surjective. □

Remark 3.14. Even for a tame system $(K;S)$ and a strictly positive invariant probability measure $\mu \in \mathrm{C}(K)'$ the epimorphism $\mathcal{K}(K;S) \longrightarrow \mathcal{J}(K,\mu;S)$ is not injective in general. In fact, if S is abelian, then so is $\mathcal{J}(K,\mu;S)$ (see Lemma 2.2 (iv)), but $\mathcal{K}(K;S)$ generally not (see [Gla07a], Example 4.5).

In Section 5 we use the semigroups introduced above to study qualitative properties (e.g., mean ergodicity) of dynamical systems.

4. Mean Ergodic Semigroups

Inspired by the approach of R. Nagel to mean ergodic semigroups (see [Nag73] and the supplement of Chapter 8 of [EFHN15]) we use techniques developed by A. Romanov in [Rom11] as well as M. Schreiber in [Sch13a] and [Sch13b] to discuss mean ergodicity of operator semigroups on locally convex spaces (see also [Ebe49], [Sat78], Section 2.1.2 in [Kre85], [GK14]; see [ABR12] for mean ergodicity of one-parameter semigroups).

Definition 4.1. Let X be a locally convex space and $\mathcal{S} \subseteq \mathscr{L}(X)$ be an operator semigroup.

(i) A net $(T_\alpha)_{\alpha\in A} \subseteq \overline{\mathrm{co}\,\mathcal{S}}^{X^X}$ is called

(a) a *left ergodic net for* $\mathcal{S}$ if

$$\lim_\alpha (\mathrm{Id} - T)T_\alpha x = 0$$

for each $x \in X$ and $T \in \mathcal{S}$.

(b) a *right ergodic net for* $\mathcal{S}$ if

$$\lim_\alpha T_\alpha(\mathrm{Id} - T)x = 0$$

for each $x \in X$ and $T \in \mathcal{S}$.

(c) a *two-sided ergodic net for* $\mathcal{S}$ if it is left and right ergodic for $\mathcal{S}$.

(ii) The semigroup $\mathcal{S}$ is called

(a) *left mean ergodic* if each left ergodic net for $\mathcal{S}$ is pointwise convergent.

(b) *right mean ergodic* if each right ergodic net for $\mathcal{S}$ is pointwise convergent.

We recall some examples (see [Sch13b], Examples 1.2).

Example 4.2. (i) Consider $\mathcal{S} = \{T^n \mid n \in \mathbb{N}_0\}$ for an operator $T \in \mathscr{L}(X)$ with bounded orbits $\mathcal{S}x$ for $x \in X$ on a locally convex space X. The sequence $(A_N)_{N\in\mathbb{N}}$ of *Cesàro means* defined by

$$A_N x := \frac{1}{N}\sum_{n=0}^{N-1} T^n x$$

for $x \in X$ and $N \in \mathbb{N}$ is a two-sided ergodic sequence for $\mathcal{S}$.

(ii) For a pointwise bounded strongly continuous semigroup $(T(t))_{t\geqslant 0}$ (i.e., $t \mapsto T(t)x$ is continuous for each $x \in X$) on a quasi-complete space X (see Definition 2.3 (iii)) we set

$$A_s x := \frac{1}{s}\int_0^s T(t)x \,\mathrm{d}t$$

for $x \in X$ and $s \in (0,\infty)$ where the integral is understood in the sense of Bourbaki (see Proposition III.3.7 in [Bou65]). Then $(A_s)_{s>0}$ is a two-sided ergodic net for the semigroup $\{T(t) \mid t \geqslant 0\}$.

(iii) The net of *Abel means* $(S_r)_{r\in(1,\infty)}$ for an operator $T \in \mathscr{L}(X)$ with bounded orbits on a quasi-complete, barrelled locally convex space X defined by

$$S_r x := (r-1)\sum_{n=0}^{\infty} \frac{1}{r^{n+1}} T^n x$$

for $x \in X$ and $r \in (1,\infty)$ is a two-sided ergodic net for $\mathcal{S} = \{T^n \mid n \in \mathbb{N}_0\}$.

(iv) Example (ii) can be generalized as follows. Consider a locally compact group G with left Haar measure μ and let $S \subseteq G$ be a subsemigroup. Assume further that there is a μ-*Følner net* $(F_\alpha)_{\alpha\in A}$ in S, i.e., $F_\alpha \subseteq S$ is a compact set with positive finite measure for each $\alpha \in A$ and

$$\lim_\alpha \frac{\mu(F_\alpha \triangle sF_\alpha)}{\mu(F_\alpha)} = 0$$

for each $s \in S$. Given a pointwise bounded representation

$$S \longrightarrow \mathscr{L}(X), \quad s \mapsto T(s)$$

of S on a quasi-complete locally convex space X we can define

$$\mathcal{F}_\alpha x := \frac{1}{\mu(F_\alpha)} \int_{F_\alpha} T(s)x \, \mathrm{d}\mu.$$

for each $x \in X$ and each $\alpha \in A$ and thereby obtain a left ergodic net for $\{T(s) \mid s \in S\}$.

Recall that a semitopological semigroup S is *left amenable* if the space $\mathrm{C_b}(S)$ of bounded continuous functions on S has a *left invariant mean*, i.e., a positive element $m \in \mathrm{C_b}(S)'$ with $m(\mathbb{1}) = 1$ and $m(L_S f) = m(f)$ for each $f \in \mathrm{C_b}(S)$ and $s \in S$, where $L_s f(t) := f(st)$ for every $t \in S$ (see Section 2.3 of [BJM89]). *Right amenability and (two-sided) amenability* are defined analogously.
We now assume that the semigroup $\mathcal{S} \subseteq \mathscr{L}(X)$ endowed with the topology of pointwise convergence is left amenable which is always the case if $\mathcal{S}$ is abelian. In this situation we can characterize the convergence of all left ergodic nets through an algebraic property of the Köhler semigroup $\mathcal{K}(\operatorname{co}\mathcal{S})$.

Theorem 4.3. *For a locally convex space X and a left amenable semigroup $\mathcal{S} \subseteq \mathscr{L}(X)$ with relatively compact convex orbits $\operatorname{co}\mathcal{S}x$ for all $x \in X$ the following are equivalent.*

(i) *The semigroup $\mathcal{S}$ is left mean ergodic.*

(ii) *The semigroup $\mathcal{K}(\operatorname{co}\mathcal{S})$ has a zero Q, i.e., $QS = SQ = Q$ for every $S \in \mathcal{K}(\operatorname{co}\mathcal{S})$.*

If these assertions hold, then $\lim T_\alpha x = Qx$ for all $x \in X$ and each left ergodic net $(T_\alpha)_{\alpha \in A}$.

Theorem 4.3 generalizes a result of A. Romanov ([Rom11]) for

$$\mathcal{S} = \{(T')^n \mid n \in \mathbb{N}_0\} \subseteq \mathscr{L}(X')$$

and a dual Banach space X' with the weak* topology. For the proof we use the methods developed by Romanov to cover our more general setting, but need some lemmas. In the first one we describe the *kernel* of $\mathcal{K}(\operatorname{co}\mathcal{S})$, i.e., the intersection of all ideals (see Notation 1.2.3 in [BJM89]).

Lemma 4.4. *Consider a locally convex space X and a left amenable semigroup $\mathcal{S} \subseteq \mathscr{L}(X)$ with relatively compact convex orbits $\operatorname{co}\mathcal{S}x$ for $x \in X$. Then the kernel of $\mathcal{K}(\operatorname{co}\mathcal{S})$ is given by*

$$\begin{aligned}\ker(\mathcal{K}(\operatorname{co}\mathcal{S})) &= \{Q \in \mathcal{K}(\operatorname{co}\mathcal{S}) \mid \mathcal{S}Q = \{Q\}\} \\ &= \{Q \in \mathcal{K}(\operatorname{co}\mathcal{S}) \mid Q \text{ is a right zero}\}.\end{aligned}$$

Proof. Set $I \coloneqq \{Q \in \mathcal{K}(\operatorname{co} \mathcal{S}) \mid \mathcal{S}Q = \{Q\}\}$ and consider the continuous mappings

$$\lambda_T \colon \mathcal{K}(\operatorname{co} \mathcal{S}) \longrightarrow \mathcal{K}(\operatorname{co} \mathcal{S}), \quad S \mapsto TS$$

for $T \in \mathcal{S}$. Since $\mathcal{S}$ is left amenable, these mappings have a common fixed point (see [Day61], Theorem 3) which yields $I \neq \emptyset$. If $Q \in I$, then clearly $\operatorname{co} \mathcal{S} = \{Q\}$ and therefore $\mathcal{K}(\operatorname{co} \mathcal{S})Q = \{Q\}$ since the semigroup is right topological. Thus, each element of I is a right zero. Moreover, I is an ideal and we thus have $\ker(\mathcal{K}(\operatorname{co} \mathcal{S})) \subseteq I$. On the other hand, each right zero Q of $\mathcal{K}(\operatorname{co} \mathcal{S})$ is a minimal idempotent and thus satisfies

$$\{Q\} = \mathcal{K}(\operatorname{co} \mathcal{S})Q \subseteq \ker(\mathcal{K}(\operatorname{co} \mathcal{S})),$$

by Theorem 1.2.12 of [BJM89]. □

Lemma 4.5. *Consider a locally convex space X and a left amenable semigroup $\mathcal{S} \subseteq \mathscr{L}(X)$ with relatively compact convex orbits $\operatorname{co} \mathcal{S}x$ for $x \in X$. For a net $(T_\alpha)_{\alpha \in A} \subseteq \mathcal{K}(\operatorname{co} \mathcal{S})$ the following are equivalent.*

(i) *The net $(T_\alpha)_{\alpha \in A}$ is left ergodic.*

(ii) *All accumulation points of $\{T_\alpha \mid \alpha \in A\}$ for the operator topology of pointwise convergence are contained in $\ker(\mathcal{K}(\operatorname{co} \mathcal{S}))$.*

Proof. Assume that $(T_\alpha)_{\alpha \in A}$ is left ergodic and take $T \in \mathcal{S}$. The map

$$\mathcal{K}(\operatorname{co} \mathcal{S}) \longrightarrow X^X, \quad S \mapsto (\mathrm{Id} - T)S$$

is continuous. Thus, for each accumulation point Q of $(T_\alpha)_{\alpha \in A}$, the operator $(\mathrm{Id} - T)Q$ is an accumulation point of $((\mathrm{Id} - T)T_\alpha)_{\alpha \in A}$. The assumption yields $(\mathrm{Id} - T)Q = 0$ and therefore $Q = TQ$. Since $T \in \mathcal{S}$ was arbitrary, we obtain $Q \in \ker(\mathcal{K}(\operatorname{co} \mathcal{S}))$ by Lemma 4.4.
Assume now that there is $T \in \mathcal{S}$ such that $((\mathrm{Id} - T)T_\alpha)_{\alpha \in A}$ does not converge to zero. We then find a zero neighborhood U and a subnet $(T_\beta)_{\beta \in B}$ of $(T_\alpha)_{\alpha \in A}$ with $(\mathrm{Id} - T)T_\beta \notin U$ for all $\beta \in B$. By compactness of $\mathcal{K}(\operatorname{co} \mathcal{S})$ we find an accumulation point Q of this subnet satisfying $(\mathrm{Id} - T)Q \neq 0$. Hence $TQ \neq Q$ and thus $(T_\alpha)_{\alpha \in A}$ has an accumulation point which is not contained in $\ker(\mathcal{K}(\operatorname{co} \mathcal{S}))$. □

Proof (of Theorem 4.3)**.** By Lemma 4.4 the second assertion is equivalent to $\ker(\mathcal{K}(\operatorname{co} \mathcal{S}))$ being the singleton $\{Q\}$ and thus Lemma 4.5 shows the implication "(ii) $\Rightarrow$ (i)".
Now suppose that (i) is valid and take $Q_1, Q_2 \in \ker(\mathcal{K}(\operatorname{co} \mathcal{S}))$. Take a family of seminorms P generating the topology on X. We define a partial order on the set

$$A \coloneqq \{(Y, M, k) \mid Y \subseteq X \text{ finite}, M \subseteq P \text{ finite}, k \in \mathbb{N}\}$$

by saying that $(Y, M, k) \leqslant (\tilde{Y}, \tilde{M}, \tilde{k})$ if $Y \subseteq \tilde{Y}$, $M \subseteq \tilde{M}$ and $k \leqslant \tilde{k}$. This order turns A into a directed set. For each triple $\alpha = (Y, M, k) \in A$ we find $T_{1,\alpha}, T_{2,\alpha} \in \operatorname{co} \mathcal{S}$ with

$$\rho(T_{i,\alpha}x - Q_i x) \leqslant \frac{1}{k}$$

for all $x \in Y$, $\rho \in M$ and $i = 1, 2$. The net given by

$$T_\alpha := \begin{cases} T_{1,\alpha} & \text{if } \alpha = (Y, M, 2n) \text{ for } n \in \mathbb{N}, \\ T_{2,\alpha} & \text{if } \alpha = (Y, M, 2n-1) \text{ for } n \in \mathbb{N}, \end{cases}$$

is left ergodic by Lemma 4.5 and hence convergent. This yields $Q_1 = Q_2$. □

We now introduce different notions of mean ergodicity on barrelled spaces. Given a pointwise bounded operator semigroup $\mathcal{S} \subseteq \mathscr{L}(X)$ on such a space, the Köhler semigroup $\mathcal{K}(\operatorname{co}\mathcal{S}'; X', X)$ for the convex hull of the adjoint semigroup

$$\mathcal{S}' = \{S' \mid S \in \mathcal{S}\}$$

is a compact right topological semigroup. To apply the results obtained above we assume that $\mathcal{S}$ is right amenable, hence $\mathcal{S}'$ is left amenable. For a net $(T_\alpha)_{\alpha \in A}$ which is right ergodic for $\mathcal{S}$ with respect to the $\sigma(X, X')$-topology the adjoint net $(T_\alpha')_{\alpha \in A}$ is left ergodic for $\mathcal{S}'$ with respect to the $\sigma(X', X)$-topology.
The following definitions are natural.

Definition 4.6. Let X be a barrelled space. A semigroup $\mathcal{S} \subseteq \mathscr{L}(X)$ is called

(i) *weak* mean ergodic* if $\mathcal{S}'$ is left mean ergodic with respect to the $\sigma(X', X)$-topology.

(ii) *weakly mean ergodic* if $\mathcal{S}$ is right mean ergodic with respect to the $\sigma(X, X')$-topology.

(iii) *strongly mean ergodic* if $\mathcal{S}$ is right mean ergodic with respect to the given topology on X.

Applying Theorem 4.3 to the $\sigma(X', X)$-topology immediately gives a characterization of weak* mean ergodicity. Next we characterize weak and strong mean ergodicity (see also Theorem 1.7 in [Nag73]) extending results of M. Schreiber for operator semigroups on Banach spaces to barrelled locally convex spaces (see Theorem 1.7 in [Sch13b], see also Corollary 1 of [Sat78] for a similar result). For a familiy $\mathcal{T}$ of operators on a locally convex space X we use the notation

$$\operatorname{fix}(\mathcal{T}) := \{x \in X \mid Tx = x \text{ for each } T \in \mathcal{T}\},$$
$$\operatorname{rg}(\mathcal{T}) := \{y \in X \mid \text{there are } x \in X \text{ and } T \in \mathcal{T} \text{ with } Tx = y\}.$$

Theorem 4.7. *Consider a bounded right amenable semigroup $\mathcal{S} \subseteq \mathscr{L}(X)$ on a barrelled locally convex space (X, τ). Then the following assertions are equivalent.*

(i) *There is a two-sided ergodic net $(T_\alpha)_{\alpha \in A}$ for $\mathcal{S}$ with respect to the weak topology such that $(T_\alpha x)_{\alpha \in A}$ converges weakly for each $x \in X$.*

(ii) *The semigroup $\mathcal{S}$ is weakly mean ergodic.*

(iii) *The semigroup $\mathcal{S}$ is strongly mean ergodic.*

(iv) *The semigroup $\mathcal{K}(\operatorname{co}\mathcal{S}; X, X')$ has a zero P.*

(v) *The semigroup* $\mathcal{K}(\operatorname{co}\mathcal{S}'; X', X)$ *has a zero* Q *which is weak* continuous.*

(vi) *The fixed space* $\operatorname{fix}(\mathcal{S})$ *separates* $\operatorname{fix}(\mathcal{S}')$.

(vii) $X = \operatorname{fix}(\mathcal{S}) \oplus \overline{\operatorname{lin}}\operatorname{rg}(\mathrm{Id} - \mathcal{S})$.

If one of the above assertions is valid, then $\lim_\alpha T_\alpha x = Px$ *in weak (resp.* τ*) topology for each operator net* $(T_\alpha)_{\alpha\in A}$ *which is right ergodic for* $\mathcal{S}$ *with respect to the weak (resp.* τ*) topology.*

We start with the following lemma.

Lemma 4.8. *Consider a bounded right amenable semigroup* $\mathcal{S} \subseteq \mathscr{L}(X)$ *on a barrelled locally convex space* (X, τ). *Then the following assertions hold.*

(i) *There exist right ergodic nets for* $\mathcal{S}$ *with respect to the topology* τ.

(ii) *Let* D *be the set of all* $x \in X$ *for which* $\lim_\alpha T_\alpha x$ *exists for every net* $(T_\alpha)_{\alpha\in A}$ *which is right ergodic with respect to* τ. *Then* D *is closed.*

(iii) *Let* D_0 *be the set of all* $x \in X$ *for which* $\lim_\alpha T_\alpha x = 0$ *for every net* $(T_\alpha)_{\alpha\in A}$ *which is right ergodic with respect to* τ. *Then* D_0 *is closed.*

(iv) $\operatorname{fix}(\mathcal{S}) \cap \overline{\operatorname{lin}}\operatorname{rg}(\mathrm{Id} - \mathcal{S}) = \{0\}$ *and* $\operatorname{fix}(\mathcal{S}) \oplus \overline{\operatorname{lin}}\operatorname{rg}(\mathrm{Id} - \mathcal{S}) \subseteq D$.

Proof. We first observe that there are right ergodic nets for $\mathcal{S}$ with respect to the $\sigma(X, X')$-topology. In fact, by Lemma 4.5 we find a left ergodic net $(T'_\alpha)_{\alpha\in A}$ for $\mathcal{S}' \subseteq \mathscr{L}(X')$ with respect to the $\sigma(X', X)$-topology such that $T'_\alpha \in \operatorname{co}\mathcal{S}'$ for all $\alpha \in A$. The net $(T_\alpha)_{\alpha\in A}$ of pre-adjoints is then right ergodic for $\mathcal{S}$ with respect to the $\sigma(X, X')$-topology.
The proof of Theorem 1.4 in [Sch13b] (which still works in the case of locally convex spaces) now shows that there are actually even right ergodic nets for $\mathcal{S}$ with respect to the topology τ.

Now if $(T_\alpha)_{\alpha\in A}$ is a right ergodic net for $\mathcal{S}$ with respect to the topology τ, then the set $\{T_\alpha \mid \alpha \in A\} \subseteq \mathscr{L}(X)$ is equicontinuous since (X, τ) is barrelled (see Theorem III.4.2 of [Sch99]). Therefore $\lim_\alpha T_\alpha x$ exists for each $x \in \overline{D}$ by Theorem III.4.5 of [Sch99] and D is closed. Similarly we see that D_0 is closed.

Since $\overline{\operatorname{lin}}\operatorname{rg}(\mathrm{Id} - \mathcal{S}) \subseteq D_0$ by (ii), we have $\operatorname{fix}(\mathcal{S}) \cap \overline{\operatorname{lin}}\operatorname{rg}(\mathrm{Id} - \mathcal{S}) = \{0\}$. Moreover, we obtain $\operatorname{fix}(\mathcal{S}) \oplus \overline{\operatorname{lin}}\operatorname{rg}(\mathrm{Id} - \mathcal{S}) \subseteq D$. □

Proof (of Theorem 4.7)**.** We first prove that (ii) implies (i). So assume (ii) and take any right ergodic net $(T_\alpha)_{\alpha\in A}$ for $\mathcal{S}$ with respect to the $\sigma(X, X')$-topology. For each $S \in \mathcal{S}$ the net $(ST_\alpha)_{\alpha\in A}$ is also right ergodic with respect to the $\sigma(X, X')$-topology. Since all right ergodic nets converge, all of them must have the same limit (otherwise the "mixed nets" would not be convergent). Thus

$$\lim_\alpha (\mathrm{Id} - S)T_\alpha x = \lim_\alpha T_\alpha x - \lim_\alpha ST_\alpha x = 0$$

weakly for each $x \in X$ and hence $(T_\alpha)_{\alpha\in A}$ is also left ergodic with respect to the $\sigma(X, X')$-topology. A similar argument shows that (iii) implies (i).

Let now $(T_\alpha)_{\alpha\in A}$ be a net as in (i). We set $Px := \lim_\alpha T_\alpha x$ for $x \in X$ where convergence is understood with respect to the weak topology $\sigma(X, X')$. Then

$P \in \mathcal{K}(\operatorname{co}\mathcal{S}; X, X')$ and P is continuous with respect to the weak topology by Proposition 2.4. Moreover, we obtain

$$0 = \lim_\alpha T_\alpha(\mathrm{Id} - T)x = Px - PTx$$
$$0 = \lim_\alpha (\mathrm{Id} - T)T_\alpha x = Px - TPx$$

for each $x \in X$ and $T \in \mathcal{S}$. This shows $PT = TP = P$ for each $T \in \mathcal{S}$ and consequently, since multiplication is separately continuous with respect to the weak operator topology,

$$PR = P = RP$$

for all $R \in \mathcal{K}(\operatorname{co}\mathcal{S}; X, X')$ and thus (iv) holds.

Suppose that (iv) is valid and let $P \in \mathcal{K}(\operatorname{co}\mathcal{S}; X, X')$ be the zero element. We then obtain $P' \in \mathcal{K}(\operatorname{co}\mathcal{S}'; X', X)$ and even $P' \in \ker \mathcal{K}(\operatorname{co}\mathcal{S}'; X', X)$ by Lemma 4.4. Now take any $Q \in \ker \mathcal{K}(\operatorname{co}\mathcal{S}'; X', X)$ and a left ergodic net $(T_\alpha')_{\alpha \in A} \subseteq \operatorname{co}\mathcal{S}'$ for $\mathcal{S}'$ with respect to the weak* topology such that $\lim_\alpha T_\alpha' = Q$. Since each operator T_α' has a pre-adjoint in $\mathcal{K}(\operatorname{co}\mathcal{S}; X, X')$ we obtain

$$Q = P'Q = \lim_\alpha P'T_\alpha' = P'$$

and hence $P' = Q$. Consequently, the kernel of $\ker \mathcal{K}(\operatorname{co}\mathcal{S}'; X', X)$ consists only of P' which shows that P' is a weak* continuous zero.

Now assume that (v) is satisfied. Let $Q = P' \in \mathcal{K}(\operatorname{co}\mathcal{S}; X'; X)$ be the weak* continuous zero and take $0 \neq x' \in \operatorname{fix}(\mathcal{S}')$. We find $x \in X$ with $\langle x, x' \rangle \neq 0$ and $y := Px \in \operatorname{fix}(\mathcal{S})$ then satisfies $\langle y, x' \rangle = \langle x, Qx' \rangle = \langle x, x' \rangle \neq 0$. Hence we have (vi).

Suppose that (vi) holds. Take $x' \in X'$ vanishing on $\operatorname{fix}(\mathcal{S}) \oplus \overline{\operatorname{lin}}\operatorname{rg}(\mathrm{Id} - \mathcal{S})$. In particular $\langle x - Sx, x' \rangle = 0$ for all $x \in X$ and $S \in \mathcal{S}$ and hence $x' \in \operatorname{fix}(\mathcal{S}')$ and $x' = 0$ since $\operatorname{fix}(\mathcal{S})$ separates $\operatorname{fix}(\mathcal{S}')$ and x' vanishes on $\operatorname{fix}(\mathcal{S})$. Thus $\operatorname{fix}(\mathcal{S}) \oplus \overline{\operatorname{lin}}\operatorname{rg}(\mathrm{Id} - \mathcal{S})$ is dense in X by the Hahn–Banach Theorem, and, by Lemma 4.8 (ii) $D = X$. Thus (vi) implies (iii).

Theorem 4.3 shows that (v) implies (ii) and therefore the equivalence of assertions (i) – (vi). The statement about the limit also follows from Theorem 4.3.

The implication "(vii) $\Rightarrow$ (iii)" is clear. Conversely, if $(T_\alpha)_{\alpha \in A}$ is a net as in (i), then $Px = \lim_\alpha T_\alpha x \in \operatorname{fix}(\mathcal{S})$ and

$$x - Px = \lim_\alpha (\mathrm{Id} - T_\alpha)x \in \overline{\operatorname{lin}}\operatorname{rg}(\mathrm{Id} - \mathcal{S}),$$

which establishes (vii). □

Corollary 4.9. *Every amenable operator semigroup $\mathcal{S} \subseteq \mathscr{L}(X)$ on a barrelled locally convex space X with relatively weakly compact convex orbits is strongly mean ergodic.*

Proof. By compactness of $\mathcal{K}(\operatorname{co}\mathcal{S}; X, X')$ the mapping

$$\Phi\colon \mathcal{K}(\operatorname{co}\mathcal{S}; X, X') \longrightarrow \mathcal{K}(\operatorname{co}\mathcal{S}'; X', X), \quad S \mapsto S'$$

is an isomorphism of right topological semigroups if we reverse the order of multiplication in $\mathcal{K}(\operatorname{co}\mathcal{S}; X, X')$. In particular, we obtain

$$\Phi(\ker(\mathcal{K}(\operatorname{co}\mathcal{S}; X, X'))) = \ker(\mathcal{K}(\operatorname{co}\mathcal{S}'; X', X)).$$

Take $P \in \ker(\mathcal{K}(\operatorname{co}\mathcal{S}; X, X'))$. Then $\mathcal{S}P = \{P\}$ and $\mathcal{S}'P' = \{P'\}$ by Lemma 4.4 and thus $SP = PS = P$ for each $S \in \mathcal{S}$. Since $\mathcal{K}(\operatorname{co}\mathcal{S}; X, X')$ is semitopological, we obtain $SP = PS = P$ for each $S \in \mathcal{K}(\operatorname{co}\mathcal{S}; X, X')$, i.e., P is a zero in $\mathcal{K}(\operatorname{co}\mathcal{S}; X, X')$. □

Remark 4.10. If X is a reflexive barrelled space, then every bounded amenable semigroup is strongly mean ergodic by Corollary 4.9 (see Theorem IV.5.6 in [Sch99]).

We present an example where Theorem 4.7 is applicable.

Example 4.11. Consider the space $\mathrm{C}(\mathbb{R})$ of continuous functions on $\mathbb{R}$ equipped with the compact-open topology, i.e., the locally convex topology induced by the seminorms ρ_K for $K \subseteq \mathbb{R}$ compact defined by

$$\rho_K(f) \coloneqq \sup_{x\in K} |f(x)|$$

for all $f \in \mathrm{C}(\mathbb{R})$. Then $\mathrm{C}(\mathbb{R})$ is a Fréchet space (and therefore barrelled) and its dual space can be identified with the compactly supported Borel measures on $\mathbb{R}$ (see Corollary 7.6.5 in [Jar81]).
Consider the multiplication operator $T \in \mathscr{L}(\mathrm{C}(\mathbb{R}))$ defined by

$$(Tf)(x) \coloneqq |\cos(x)| \cdot f(x)$$

for each $f \in \mathrm{C}(\mathbb{R})$ and each $x \in \mathbb{R}$. Then $\mathcal{S} \coloneqq \{T^n \mid n \in \mathbb{N}_0\}$ is bounded. Moreover we have $\operatorname{fix}(\mathcal{S}) = \{0\}$ and $\operatorname{lin}\{\delta_{\pi k} \mid k \in \mathbb{Z}\} \subseteq \operatorname{fix}(\mathcal{S}')$. Thus $\mathcal{S}$ is not strongly mean ergodic by Theorem 4.7. However, it is weak* mean ergodic. In fact, for each $f \in \mathrm{C}(\mathbb{R})$ we obtain $\lim_{n\to\infty} T^n f = Pf$ pointwise with

$$(Pf)(x) \coloneqq \begin{cases} f(\pi k) & \text{if } x = \pi k \text{ with } k \in \mathbb{Z}, \\ 0 & \text{else.} \end{cases}$$

Lebesgue's Theorem implies

$$\lim_{n\to\infty} \langle f, (T')^n \mu\rangle = \int_{\mathbb{R}} Pf \,\mathrm{d}\mu = \langle f, \sum_{k\in\mathbb{Z}} \mu(\{\pi k\})\delta_{\pi k}\rangle$$

for each $f \in \mathrm{C}(\mathbb{R})$ and each $\mu \in \mathrm{C}(\mathbb{R})'$. Thus $\lim_{n\to\infty}(T')^n\mu = \sum_{k\in\mathbb{Z}} \mu(\pi k)\delta_{\pi k}$ in weak* topology which implies

$$\lim_{\alpha} S_\alpha \mu = \sum_{k\in\mathbb{Z}} \mu(\{\pi k\})\delta_{\pi k}$$

in weak* topology for each $\mu \in \mathrm{C}(\mathbb{R})'$ and each left ergodic net $(S_\alpha)_{\alpha\in A}$ for $\mathcal{S}'$.

5. Mean Ergodicity in Topological Dynamics

In this section we study different notions of mean ergodicity in topological dynamics (see [Sch14]). We note that for a topological dynamical system $(K; S)$ the mapping

$$S \longrightarrow \mathcal{T}_S, \quad s \mapsto T_s$$

is an epimorphism of semitopological semigroups, if we reverse the order of multiplication in S and equip $\mathcal{T}_S$ with the strong operator topology (see Theorem 4.17 in [EFHN15]). In particular, if S is left amenable, then $\mathcal{T}_S$ is right amenable.

A topological dynamical system $(K;S)$ is said to be *weak** (resp. *norm*) *mean ergodic* if the Koopman semigroup $\mathcal{T}_S \subseteq \mathscr{L}(\mathrm{C}(K))$ is weak* (resp. strongly) mean ergodic. Moreover, the system $(K;S)$ is *uniquely ergodic* if there is a unique S-invariant probability measure $\mu \in \mathrm{C}(K)'$. Using the convex Köhler semigroup (see Definition 3.4) we obtain the following characterization of unique ergodicity.

Proposition 5.1. *Let $(K;S)$ be a topological dynamical system with S left amenable. The following assertions are equivalent.*

(i) *There is a net $(T_\alpha)_{\alpha\in A} \subseteq \mathrm{co}\,\mathcal{T}_S$ for $\mathcal{T}_S$ which is right ergodic with respect to the weak topology such that for each $f \in \mathrm{C}(K)$ the net $(T_\alpha f)_{\alpha\in A}$ converges weakly to a constant function.*

(ii) *The system $(K;S)$ is uniquely ergodic.*

(iii) *The semigroup $\mathcal{K}_\mathrm{c}(K;S)$ has a zero which is a rank one operator.*

If one of these assertions holds and $\mu \in \mathrm{C}(K)'$ is the unique invariant probability measure, then

$$\lim_\alpha T_\alpha f = \int_K f \,\mathrm{d}\mu \cdot \mathbb{1}$$

uniformly on K for each $f \in \mathrm{C}(K)$ and for each net $(T_\alpha)_{\alpha\in A}$ which is right ergodic for $\mathcal{T}_S$ with respect to the norm topology.

Proof. Assume that $(T_\alpha)_{\alpha\in A}$ is a net as in (i). For each $f \in \mathrm{C}(K)$ let $c(f) \in \mathbb{C}$ with $\lim_{n\to\infty} T_\alpha f = c(f)\cdot\mathbb{1}$. Then

$$\lim_\alpha (\mathrm{Id} - T_s)T_\alpha f = c(f)\cdot\mathbb{1} - c(f)\cdot T_s\mathbb{1} = 0$$

weakly for each $f \in \mathrm{C}(K)$ and therefore $(T_\alpha)_{\alpha\in A}$ is also left ergodic. Thus assertion (i) of Theorem 4.7 holds. By (v) of Theorem 4.7 we obtain that $\mathrm{fix}(\mathcal{T}_S)$ separates $\mathrm{fix}(\mathcal{T}_S')$. But for each $f \in \mathrm{fix}(\mathcal{T}_S)$ we have

$$f = \lim_{n\to\infty} T_\alpha f = c(f)\cdot\mathbb{1},$$

hence $\mathrm{fix}(\mathcal{T}_S)$ is one dimensional and so must $\mathrm{fix}(\mathcal{T}_S')$ proving (ii).

Suppose that (ii) is valid and let $Q_1, Q_2 \in \ker(\mathcal{K}_\mathrm{c}(K;S))$. For each probability meausure $\mu \in \mathrm{C}(K)'$ the measures $Q_1\mu, Q_2\mu \in \mathrm{C}(K)'$ are invariant probability measures and thus $Q_1\mu = Q_2\mu$. This implies $Q_1 = Q_2$ and therefore $\mathcal{K}_\mathrm{c}(K;S)$ has a zero Q. If $\mu_1, \mu_2 \in \mathrm{C}(K)'$ are two probability measures, we also obtain $Q\mu_1 = Q\mu_2$. As a result Q has rank one.

Finally assume (iii). Let $Q \in \mathcal{K}_\mathrm{c}(K;S)$ be the zero which is a rank one operator. Take $x \in K$ and set $\mu \coloneqq Q\delta_x$. Since Q is rank one, we obtain $Q\nu = Q\delta_x = \mu$ for each probability measure $\nu \in \mathrm{C}(K)'$. Now consider the operator $P \in \mathscr{L}(\mathrm{C}(K))$ given by

$$Pf \coloneqq \langle f, \mu\rangle \cdot \mathbb{1}$$

for $f \in \mathrm{C}(K)$. We then obtain

$$\langle Pf, \nu\rangle = \langle f, \mu\rangle \cdot \langle \mathbb{1}, \nu\rangle = \langle f, \mu\rangle = \langle f, Q\nu\rangle$$

for each $f \in \mathrm{C}(K)$ and each probability measure $\nu \in \mathrm{C}(K)'$. Hence $P' = Q$ and Q is weak* continuous. Thus (i) and the remaining assertion follow from Theorem 4.7. □

Remark 5.2. The equivalence of (i) and (ii) is also a direct consequence of Theorem 1.7 of [Sch13b]. The new part of Proposition 5.1 is the characterization of unique ergodicity via properties of the zero $Q \in \mathcal{K}_\mathrm{c}(K;S)$. In fact, we have proved the following for topological dynamical systems $(K;S)$ with S left amenable.

(i) $(K;S)$ is weak* mean ergodic if and only if $\mathcal{K}_\mathrm{c}(K;S)$ has a zero (see Theorem 4.3).

(ii) $(K;S)$ is norm mean ergodic if and only if $\mathcal{K}_\mathrm{c}(K;S)$ has a weak* continuous zero (see Theorem 4.7).

(iii) $(K;S)$ is uniquely ergodic if and only if $\mathcal{K}_\mathrm{c}(K;S)$ has a zero which is a rank one operator (see Proposition 5.1).

Recall that a topological dynamical system $(K;S)$ is *minimal* if K has no non-trivial closed S-invariant subsets. The following consequence of Proposition 5.1 is a variation of [KW81], Proposition 3.2, for two-sided ergodic sequences (see also the remark below Corollary 3.3 in [Rom11] and the paper by Iwanik [Iwa80]).

Corollary 5.3. *Consider a minimal topological dynamical system $(K;S)$ with S left amenable. If there is an operator sequence $(T_n)_{n\in\mathbb{N}} \subseteq \operatorname{co}\mathcal{T}_S$ which is two-sided ergodic for $\mathcal{T}_S$ with respect to the $\sigma(\mathrm{C}(K),\ell^1(K))$-topology such that $(T_nf)_{n\in\mathbb{N}}$ converges pointwise for each $f \in \mathrm{C}(K)$, then $(K;S)$ is uniquely ergodic.*

Proof. Take a sequence $(T_n)_{n\in\mathbb{N}}$ as above and set $Pf(x) := \lim_{n\to\infty} T_nf(x)$ for $x \in K$ and $f \in \mathrm{C}(K)$. Then P maps $\mathrm{C}(K)$ to the space of Baire 1 functions $\mathrm{B}_1(K)$.
For $f \in \mathrm{C}(K)$ and $x_1, x_2 \in K$ the pre-images $M_i := (Pf)^{-1}(Pf(x_i))$ are non-empty, S-invariant G_δ sets for $i = 1, 2$ and—by minimality of $(K;S)$—dense. In Baire spaces the intersection of two dense G_δ sets is dense and in particular non-empty. We conclude $Pf(x_1) = Pf(x_2)$ and therefore Pf is constant. By Lebesgue's Theorem and Proposition 5.1 $(K;S)$ is uniquely ergodic. □

Corollary 5.4. *Every minimal tame metric topological dynamical system $(K;S)$ with S amenable is uniquely ergodic.*

Proof. By Proposition 1.3 of [Sch13b] there exists a two-sided ergodic net $(T_\alpha)_{\alpha\in A}$ with respect to the weak topology $\sigma(\mathrm{C}(K),\mathrm{C}(K)')$. By passing to a subnet we may assume that the limit $Q\mu := \lim_\alpha T'_\alpha\mu$ exists in weak* topology for each $\mu \in \mathrm{C}(K)'$. Then $Q \in \ker\mathcal{K}_\mathrm{c}(K;S)$. Since $(K;S)$ is tame, $\mathcal{K}_\mathrm{c}(K;S)$ is a Rosenthal compact space and we find a left ergodic sequence $(S'_n)_{n\in\mathbb{N}} \subseteq \operatorname{co}\mathcal{T}'_S$ converging to Q. Since $QT'_s = T'_sQ = Q$ for each $s \in S$, the sequence $(S'_n)_{n\in\mathbb{N}}$ is also two-sided ergodic. In particular $(S_n)_{n\in\mathbb{N}}$ is a two-sided ergodic sequence for $\mathcal{T}_S$ with respect to the $\sigma(\mathrm{C}(K),\ell^1(K))$-topology such that $(S_nf)_{n\in\mathbb{N}}$ converges pointwise for each $f \in \mathrm{C}(K)$ and hence $(K;S)$ is uniquely ergodic by Corollary 5.3. □

Remark 5.5. Corollary 5.4 has been proved for abelian group actions by E. Glasner (see Theorem 5.1 in [Gla07b]), D. Kerr and H. Li (see Theorem 7.19 [KL07]) as well as W. Huang (see Theorem 4.8 in [Hua06]). Their proofs are based on a representation type result for minimal tame systems while our proof uses the topological properties of the semigroup $\mathcal{K}_{\mathrm{c}}(K;S)$.

The following theorem is the main result of this section. We recall that a topological dynamical system $(K;S)$ is *topologically transitive* if there is $x \in K$ such that $\overline{Sx} = K$.

Theorem 5.6. *Consider a topological dynamical system $(K;S)$ with S left amenable and the following assertions.*

(i) *$(K;S)$ is weak* mean ergodic.*

(ii) *$(K;S)$ is norm mean ergodic.*

(iii) *$(K;S)$ is uniquely ergodic.*

Then (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i). If $(K;S)$ is topologically transitive, all these assertions are equivalent.

Remark 5.7. Simple examples show that the three notions of weak* mean ergodicity, norm mean ergodicity and unique ergodicity are truly distinct.

For the proof of Theorem 5.6 we need two lemmas. We write $\mathrm{P}(K)$ for the probability measures on a compact space K and remind the reader that we identify K with the space of Dirac measures $\{\delta_x \mid x \in K\} \subseteq \mathrm{P}(K)$.

Lemma 5.8. *Consider a topological dynamical system $(K;S)$ with S left amenable and a point $x \in K$. Then the following identities hold.*

(i) $\mathcal{K}(K;S)(x) = \overline{Sx} \subseteq K$.

(ii) $\mathcal{K}_{\mathrm{c}}(K;S)(x) = \mathrm{P}(\overline{Sx}) \subseteq \mathrm{P}(K)$.

Proof. Since $\mathcal{K}(K;S)$ is compact, assertion (i) is obvious. For (ii) we obtain, by the Krein–Milman theorem,

$$\mathrm{P}(\overline{Sx}) = \overline{\mathrm{co}\,\{\delta_y \mid y \in \overline{Sx}\}} = \overline{\mathrm{co}\,\{\delta_{sx} \mid s \in S\}} = \mathcal{K}_{\mathrm{c}}(K;S)(x),$$

where the latter equation is a consequence of the compactness of $\mathcal{K}_{\mathrm{c}}(K;S)$. □

Lemma 5.9. *Consider a weak* mean ergodic topological dynamical system $(K;S)$ with S left amenable. Then for each $x \in K$ the orbit system $(\overline{Sx};S)$ is uniquely ergodic.*

Proof. Since subsystems of weak* mean ergodic systems are again weak* mean ergodic, we may assume—by passing to an orbit system—that $(K;S)$ is transitive. Take $x \in K$ with $K = \overline{Sx}$ and let $Q \in \mathcal{K}_{\mathrm{c}}(K;S)$ be the zero element. The measure $\mu := Qx \in \mathrm{C}(K)'$ is S-invariant.
Now consider an invariant probability measure $\nu \in \mathrm{C}(K)'$. By Lemma 5.8 we find an operator $T \in \mathcal{K}_{\mathrm{c}}(K;S)$ with $\nu = Tx$. This yields

$$\nu = Q\nu = QTx = Qx = \mu.$$

□

Proof (of Theorem 5.6)**.** It is obvious that (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i). If $(K;S)$ is transitive, then Lemma 5.9 proves the equivalence. $\square$

To conclude this section we characterize weak* and "pointwise" mean ergodicity for tame metric systems. Our theorem extends Theorem 4.5 of [Rom16] where a similar result was shown for $\mathbb{N}_0$-actions with metrizable Ellis semigroup (called *ordinary systems*).

Theorem 5.10. *Consider a tame metric topological dynamical system $(K;S)$ with S amenable. Then the following assertions are equivalent.*

(i) *For each operator net $(T_\alpha)_{\alpha\in A}$ which is right ergodic for $\mathcal{T}_S$ with respect to the $\sigma(\mathrm{C}(K), \ell^1(K))$-topology, each $f \in \mathrm{C}(K)$ and each $x \in K$ the limit $\lim_\alpha (T_\alpha f(x))_{\alpha\in A}$ exists.*

(ii) *The system $(K;S)$ is weak* mean ergodic.*

(iii) *For each $x \in X$ the system $(\overline{Sx};S)$ contains a unique minimal set.*

Once again we need two lemmas. The first one is a generalization of Lemma 2.3 of [Rom11] to our setting.

Lemma 5.11. *Let $(K;S)$ be a topological dynamical system and consider $\psi \in \ker \mathrm{E}(K;S)$. Then $\psi(K)$ is contained in the union of minimal sets.*

Proof. By Theorem 1.2.12 in [BJM89] we find a minimal left ideal I of $\mathrm{E}(K;S)$ containing ψ. However, the set $I(x)$ is minimal by Proposition 1.6.12 in [BJM89]. $\square$

We need a more general version of Corollary 5.4.

Lemma 5.12. *Let $(K;S)$ be a tame metric topological dynamical system with S amenable containing a unique minimal subset. Then $(K;S)$ is uniquely ergodic.*

Proof. Denote the unique minimal subset by M and consider an invariant probability measure $\mu \in \mathrm{C}(K)'$. Since the support of μ is closed and invariant, it contains M.
Now take a minimal idempotent $\psi = \lim_{n\to\infty} s_n \in \mathrm{E}(K;S)$. Then $\psi(K) \subseteq M$ by Lemma 5.11 and for each positive $f \in \mathrm{C}(K)$ vanishing on M we obtain

$$\langle f, \mu\rangle = \lim_{n\to\infty} \langle f, T_{s_n}'\mu\rangle = \langle f, T_\psi' \mu\rangle = \langle T_\psi f, \mu\rangle = 0.$$

This shows $\operatorname{supp}\mu = M$ and Corollary 5.4 proves the claim. $\square$

Before proceeding to the proof of Theorem 5.10, we observe that pointwise and weak ergodic nets are the same for tame systems.

Lemma 5.13. *Consider a tame metric topological dynamical $(K;S)$ with S amenable. Each operator net $(T_\alpha)_{\alpha\in A}$ which is right ergodic for $\mathcal{T}_S$ with respect to the $\sigma(\mathrm{C}(K), \ell^1(K))$-topology is right ergodic for $\mathcal{T}_S$ with respect to the $\sigma(\mathrm{C}(K), \mathrm{C}(K)')$-topology.*

Proof. Take an operator net $(T_\alpha)_{\alpha\in A}$ which is right ergodic with respect to the $\sigma(\mathrm{C}(K), \ell^1(K))$-topology, $f \in \mathrm{C}(K)$ and $s \in S$. We then have $\lim_\alpha (T_\alpha(f - T_s f))(x) = 0$ for all $x \in K$.
Equip $\mathrm{B}_1(K)$ with the topology of pointwise convergence. Since $(K;S)$ is

tame, the set $\overline{\mathrm{co}\, \mathcal{T}_S(\mathrm{Id} - T_s) f}$ is compact in $\mathrm{B}_1(K)$ and it contains the net $(T_\alpha(\mathrm{Id} - T_s)f)_{\alpha \in A}$. The main theorem of [Ros77] therefore implies

$$\lim_\alpha T_\alpha(\mathrm{Id} - T_s)f = 0$$

with respect to the $\sigma(\mathrm{C}(K), \mathrm{C}(K)')$-topology. □

Proof (of Theorem 5.10)**.** If $(K;S)$ is weak* mean ergodic, then each orbit is uniquely ergodic by Lemma 5.9 and—since every minimal set supports an invariant probability measure—contains only one minimal set. This proves the implication "(ii) $\Rightarrow$ (iii)".

By Lemma 5.12 assertion (iii) implies that each orbit is uniquely ergodic. Thus, for each right $\sigma(\mathrm{C}(K), \mathrm{C}(K)')$-ergodic net $(T_\alpha)_{\alpha \in A}$ for $\mathcal{T}_S$ we obtain that $(T_\alpha f)_{\alpha \in A}$ converges weakly and thus pointwise on each orbit. But then $(T_\alpha f)_{\alpha \in A}$ converges pointwise on K. Combined with Lemma 5.13 this implies (i).

Finally suppose that (i) holds and take two elements $Q_1, Q_2 \in \ker(\mathcal{K}_{\mathrm{c}}(K;S))$. Since $\mathcal{K}_{\mathrm{c}}(K;S)$ is a Fréchet–Urysohn space, we find right ergodic sequences $(T_{i,n}')_{n \in \mathbb{N}} \subseteq \mathrm{co}\, \mathcal{T}_S'$ for $\mathcal{T}_S'$ converging to Q_i for $i = 1, 2$. Consider the sequences $(T_{i,n})_{n \in \mathbb{N}}$ consisting of the pre-adjoints and observe that for each $f \in \mathrm{C}(K)$ we have $\lim_{n \to \infty} T_{i,n} f = Q_i' f$ with respect to the $\sigma(\mathrm{C}(K)'', \mathrm{C}(K)')$-topology for $i = 1, 2$. By assumption the sequence obtained by alternating the members of $(T_{1,n} f)_{n \in \mathbb{N}}$ and $(T_{2,n} f)_{n \in \mathbb{N}}$ converges pointwise and thus, by Lebegue's Theorem, in the $\sigma(\mathrm{C}(K)'', \mathrm{C}(K)')$-topology. This yields $Q_1 = Q_2$. □

The next example shows that even for tame systems the weak* convergence of a single ergodic sequence does not ensure weak* mean ergodicity.

Example 5.14. Consider the space $\{0,1\}^{\mathbb{N}}$ with the product topology (which is compact and metrizable) and endow it with the shift φ given by $\varphi\left((a_n)_{n \in \mathbb{N}}\right) := (a_{n+1})_{n \in \mathbb{N}}$ for $(a_n)_{n \in \mathbb{N}} \in \{0,1\}^{\mathbb{N}}$. Consider the point $x = (x_n)_{n \in \mathbb{N}} \in \{0,1\}^{\mathbb{N}}$ with

$$x_n = \begin{cases} 1 & \text{if } n \in \{k(N)+1, ..., k(N)+N\}, \\ 0 & \text{if } n \in \{k(N)+N+1, ..., k(N+1)\}, \end{cases}$$

where

$$k(N) := \sum_{n=1}^{N-1} (n + 10^n) = \frac{N \cdot (N-1)}{2} + 10 \cdot \frac{10^{N-1} - 1}{9}$$

for $N \in \mathbb{N}$. For illustration we give the start of this sequence as

$$x = (1, \underbrace{0, ..., 0}_{10 \text{ zeroes}}, 1, 1, \underbrace{0, ..., 0}_{100 \text{ zeroes}}, 1, 1, 1, \underbrace{0, ..., 0}_{1000 \text{ zeroes}}, 1, 1, 1, 1, 0, ...).$$

Now consider the compact subspace $K := \overline{\{\varphi^n(x) \mid n \in \mathbb{N}_0\}}$ and the system $(K;S)$ with $S := \{(\varphi|_K)^n \mid n \in \mathbb{N}_0\}$. It is easy to see that K is countable and thus $\mathrm{E}(K;S) \subseteq K^K$ has cardinality at most $\mathfrak{c}$. Therefore the system is tame by Theorem 1.2 of [Gla06].

For each $f \in \mathrm{C}(K)$ the Cesàro means $(\frac{1}{N} \sum_{n=0}^{N-1} T_\varphi^n f)_{n \in \mathbb{N}}$ converge pointwise (and therefore with respect to the weak* topology) to the function

$Pf\colon K \longrightarrow \mathbb{C}$ with

$$Pf((x_n)_{n\in\mathbb{N}}) := \begin{cases} f((1)_{n\in\mathbb{N}}) & \text{if there is } N \in \mathbb{N} \text{ with } x_n = 1 \text{ for all } n \geqslant N, \\ f((0)_{n\in\mathbb{N}}) & \text{else.} \end{cases}$$

On the other hand, the constant zero sequence and the constant one sequence are two fixed points of the system. Thus $(K;S)$ is not weak* mean ergodic by Theorem 5.10.

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1.2 The primitive spectrum of a semigroup of Markov operators

THE PRIMITIVE SPECTRUM OF A SEMIGROUP OF MARKOV OPERATORS

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Abstract. For a semigroup $\mathcal{S}$ of Markov operators on a space of continuous functions, we use $\mathcal{S}$-invariant ideals to describe qualitative properties of $\mathcal{S}$ such as mean ergodicity and the structure of its fixed space. For this purpose we focus on *primitive $\mathcal{S}$-ideals* and endow the space of those ideals with an appropriate topology. This approach is inspired by the representation theory of C*-algebras and can be adapted to our dynamical setting.
In the particularly important case of Koopman semigroups, we characterize the centers of attraction of the underlying dynamical system in terms of the invariant ideal structure of $\mathcal{S}$.

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1. Introduction

The *primitive spectrum* is a useful tool in the study of C*-algebras (see, e.g., Chapter IV of [Dix77], Section 4.3 of [Ped79] or Section II.6.5 of [Bla06]) and plays a crucial role in representation theory (cf. [Hof11]). Given a C*-algebra A it is defined as

$$\mathrm{Prim}(A) := \{\ker \pi \mid 0 \neq \pi \text{ irreducible representation of A}\}.$$

Equipped with the *hull-kernel topology* (also called *Jacobson topology*) it becomes a quasi-compact T_0-space. A nice application is the so called Dauns-Hofmann Theorem asserting that—in the unital case—the center of A is canonically isomorphic to $\mathrm{C}(\mathrm{Prim}(A))$.
In this note we study a dynamical version of the primitive spectrum in the commutative and unital case. Starting from a right amenable semigroup $\mathcal{S}$ of Markov operators on the space of continuous functions $\mathrm{C}(K)$ on some compact space K we introduce the primitive spectrum $\mathrm{Prim}(\mathcal{S})$ of $\mathcal{S}$ as the set of absolute kernels of ergodic measures. Again we equip the primitive spectrum with a hull-kernel topology and obtain a quasi-compact T_0-space. We then describe the space $\mathrm{C}(\mathrm{Prim}(\mathcal{S}))$ and give applications to topological dynamics and ergodic theory.

We now give a more detailed description of the results.
Based on two papers of H. H. Schaefer (see [Sch67] and [Sch68]) as well as Paragraph III.8 of [Sch74] we consider $\mathcal{S}$-invariant ideals and measures in

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Section 2 recalling some basic definitions and facts.
In the subsequent sections we introduce and study radical $\mathcal{S}$-ideals. In Section 3 we give the definition and prove an equivalent characterization in the metric case (see Proposition 3.7). In the fourth section we then establish a close connection between radical $\mathcal{S}$-ideals, centers of attraction appearing in topological dynamics and stability conditions of the semigroup (see Theorem 4.2 and Theorem 4.9).
The last three sections are devoted to the primitive spectrum of $\mathcal{S}$ as a topological space and its applications. In Section 5 we define the topology (cf. Proposition 5.3), state its basic properties and give some examples.
In Section 6 we then prove a Dauns-Hofmann-type theorem showing that if $\mathcal{S}$ is radical free (see Definition 3.1) the space of continuous functions on the primitive spectrum $\mathrm{C}(\mathrm{Prim}(\mathcal{S}))$ is canonically isomorphic to the fixed space $\mathrm{fix}(\mathcal{S})$ of the semigroup $\mathcal{S}$ (see Theorem 6.3). We then extend this result to the general case of not necessarily radical free $\mathcal{S}$ (see Theorem 6.5).
As an application we obtain in Section 7 a new description of mean ergodicity of semigroups of Markov operators (see Theorem 7.1) which generalizes Schaefer's Theorem 2 of [Sch67] in two different ways. On one hand we consider the more general setting of right amenable semigroups instead of single operators. But more importantly, we obtain—in contrast to Schaefer's work—a full characterization of mean ergodicity . We finally look at some examples illustrating these results (cf. Examples 7.5).
It should be pointed out that while maximal invariant ideals (which have been the central objects in [Sch67] and [Sch68]) are enough to describe mean ergodic Markov operators and semigroups, the results of our paper show that primitive ideals are the natural algebraic structure to describe dynamical properties of general Markov semigroups, see also Remark 7.2 below.

In the following we always assume K to be a compact (Hausdorff) space. We denote the Banach lattice of continuous complex-valued functions on K by $\mathrm{C}(K)$ and identify the dual space $\mathrm{C}(K)'$ of $\mathrm{C}(K)$ with the Banach lattice of complex regular Borel masures on K. Moreover, we refer to [Sch74] and [MN91] for Banach lattices and their ideal structure and remind the reader that the closed lattice ideals of $\mathrm{C}(K)$ coincide with the closed algebra ideals and are precisely the sets

$$I_L := \{f \in \mathrm{C}(K) \mid f|_L = 0\}$$

with $L \subseteq K$ closed. Recall also that a positive operator $T \in \mathscr{L}(\mathrm{C}(K))$ is called *Markov* if $T\mathbb{1} = \mathbb{1}$.
We now fix a semigroup $\mathcal{S} \subseteq \mathscr{L}(\mathrm{C}(K))$ of Markov operators which is *right amenable* (cf. Section 2.3 of [BJM89]) if endowed with the strong operator topology, i.e., there is a positive element $m \in \mathrm{C_b}(\mathcal{S})'$ (called *right invariant mean*) such that $m(\mathbb{1}) = 1$ and $m(R_S f) = m(f)$ for every $f \in \mathrm{C_b}(\mathcal{S})$ where $R_S(f)(T) := f(TS)$ for all $T, S \in \mathcal{S}$. All abelian topological semigroups and compact topological groups are amenable and, in particular, right amenable. For more examples and counterexamples we refer to [Day61] and Chapter 1 of [Pat88].
Note that the important cases of semigroups generated by a single operator and one-parameter semigroups are contained in our results since these are

always abelian.
Many examples of Markov operators and semigroups arise from topological dynamical systems on K. In fact, if $\varphi\colon K \longrightarrow K$ is a continuous mapping, then the asscoiated *Koopman operator* $T_\varphi \in \mathscr{L}(\mathrm{C}(K))$ defined by $f := f \circ \varphi$ for $f \in \mathrm{C}(K)$ is a Markov lattice operator. We write $\mathcal{S}_\varphi$ for the semigroup $\{T_\varphi^n \mid n \in \mathbb{N}_0\}$.

2. Ergodic Measures and Primitive Ideals

In this section we introduce primitive $\mathcal{S}$-ideals adapting concepts from the theory of C*-algebras and start with the following definition going back to H. H. Schaefer (see [Sch67]). Recall that an ideal I of $\mathrm{C}(K)$ is called *proper* if $I \neq \mathrm{C}(K)$.

Definition 2.1. A closed proper ideal $I \subseteq \mathrm{C}(K)$ is an $\mathcal{S}$*-ideal* if it is $\mathcal{S}$-invariant, i.e., $SI \subseteq I$ for each $S \in \mathcal{S}$. It is called *maximal* if it is maximal among all $\mathcal{S}$-ideals with respect to inclusion.

Remark 2.2. If I is an $\mathcal{S}$-ideal, then a standard application of Zorn's lemma shows that I is contained in a maximal proper $\mathcal{S}$-invariant ideal. Since there are no dense proper ideals in $\mathrm{C}(K)$, this ideal is already closed and therefore each $\mathcal{S}$-ideal is contained in a maximal $\mathcal{S}$-ideal (cf. Proposition 1 in [Sch67]).

In [Sin68] R. Sine used the concept of a self-supporting set of a Markov operator. Generalizing this to our setting, a non-empty closed set $L \subseteq K$ is called *self-supporting* if the measure $S'\delta_x \in \mathrm{C}(K)'$ has support in L for each $x \in L$ and $S \in \mathcal{S}$. Recall that here the *support* $\operatorname{supp}\mu$ of a probability measure $\mu \in \mathrm{C}(K)'$ is the smallest closed subset $A \subseteq K$ with $\mu(A) = 1$.
Each self-supporting set L defines an $\mathcal{S}$-ideal

$$I_L := \{f \in \mathrm{C}(K) \mid f|_L = 0\}.$$

Conversely, each $\mathcal{S}$-ideal is an I_L for some self-supporting set L and the mapping $L \mapsto I_L$ is bijective. Moreover, each maximal $\mathcal{S}$-ideal corresponds to a minimal self-supporting set.
Given an $\mathcal{S}$-ideal I we call the unique self-supporting set L with $I_L = I$ the *support of* I and write $L = \operatorname{supp} I$.

Remark 2.3. For each $\mathcal{S}$-ideal I, the semigroup $\mathcal{S}$ induces a semigroup $\mathcal{S}_I$ of Markov operators on $\mathrm{C}(\operatorname{supp} I)$ given by

$$\mathcal{S}_I := \{S_I \mid S \in \mathcal{S}\}$$

with $S_I f := SF|_{\operatorname{supp} I}$ for $S \in \mathcal{S}$ and $f \in \mathrm{C}(\operatorname{supp} I)$ where $F \in \mathrm{C}(K)$ is any extension of f to K. It is readily checked that I is maximal if and only if $\mathcal{S}_I$ is irreducible, i.e., there are no non-trivial $\mathcal{S}_I$-ideals (see the corollary to Proposition III.8.2 in [Sch74]).

We are primarily interested in $\mathcal{S}$-ideals defined by measures. The *absolute kernel* of a measure $0 \leqslant \mu \in \mathrm{C}(K)'$ is

$$I_\mu := \{f \in \mathrm{C}(K) \mid \langle |f|, \mu\rangle = 0\}.$$

If μ is invariant, i.e., $S'\mu = \mu$ for each $S \in \mathcal{S}$, this is an $\mathcal{S}$-ideal.
We write $\mathrm{P}_{\mathcal{S}}(K) \subseteq \mathrm{C}(K)'$ for the space of invariant probability measures on K equipped with the weak* topology. By right amenability of $\mathcal{S}$ this

is always a nonempty compact convex set (this is a simple consequence of Day's fixed point theorem, see Theorem 3 of [Day61]).

We recall that for each $\mu \in \mathrm{P}_{\mathcal{S}}(K)$ the space $\mathrm{L}^1(K,\mu)$ is the completion of $\mathrm{C}(K)/I_\mu$ with respect to the L^1-norm. Since I_μ is $\mathcal{S}$-invariant, every $S \in \mathcal{S}$ induces an operator on $\mathrm{C}(K)/I_\mu$ which then uniquely extends to a bi-Markov operator S_μ on $\mathrm{L}^1(K,\mu)$, i.e., S_μ is a positive operator on $\mathrm{L}^1(K,\mu)$ with $S_\mu \mathbb{1} = \mathbb{1}$ and $S_\mu' \mathbb{1} = \mathbb{1}$. We write $\mathcal{S}_\mu := \{S_\mu \mid S \in \mathcal{S}\}$ for the semigroup on $\mathrm{L}^1(K,\mu)$ induced by $\mathcal{S}$.

Definition 2.4. A measure $\mu \in \mathrm{P}_{\mathcal{S}}(K)$ is called *ergodic* if the fixed space $\mathrm{fix}(\mathcal{S}_\mu)$ in $\mathrm{L}^1(K,\mu)$ is one-dimensional.

The following characterization of ergodicity generalizes a result of M. Rosenblatt (cf. [Ros76]) and is well-known for single operators. We give a short proof in case of semigroup actions inspired by the proof of Proposition 10.4 of [EFHN15]. Here and in the following we write $\mathrm{ex}\, M$ for the set of extreme points of a convex subset M of a vector space.

Proposition 2.5. *A measure $\mu \in \mathrm{P}_{\mathcal{S}}(K)$ is ergodic if and only if $\mu \in \mathrm{ex}\,\mathrm{P}_{\mathcal{S}}(K)$.*

Proof. Assume that $\mathrm{fix}(\mathcal{S}_\mu)$ is not one-dimensional. Since $\mathrm{fix}(\mathcal{S}_\mu)$ is an AL-sublattice of $\mathrm{L}^1(K,\mu)$ with weak order unit $\mathbb{1}$, the set

$$B = \{f \in \mathrm{fix}(\mathcal{S}_\mu) \mid f \geqslant 0 \text{ and } \sup(f, \mathbb{1} - f) = 0\}$$

is total in $\mathrm{fix}(\mathcal{S}_\mu)$ (cf. page 115 of [Sch74]). But B is just the set of characteristic functions in $\mathrm{fix}(\mathcal{S}_\mu)$. Thus there is a measurable set $A \subseteq K$ with $S_\mu \mathbb{1}_A = \mathbb{1}_A$ and $0 < \mu(A) < 1$. Now consider the measures μ_1, μ_2 defined by

$$\mu_1(g) := \frac{1}{\mu(A)} \int_A g \,\mathrm{d}\mu \text{ and } \mu_2(g) := \frac{1}{1-\mu(A)} \int_{K\setminus A} g\,\mathrm{d}\mu$$

for $g \in \mathrm{C}(K)$. For every $g \in \mathrm{C}(K)$ with $0 \leqslant g \leqslant \mathbb{1}$ and each $S \in \mathcal{S}$ we obtain

$$\int_A g\,\mathrm{d}\mu = \int g \wedge \mathbb{1}_A \,\mathrm{d}\mu = \int S_\mu(g \wedge \mathbb{1}_A)\,\mathrm{d}\mu \leqslant \int Sg \wedge \mathbb{1}_A \,\mathrm{d}\mu = \int_A Sg\,\mathrm{d}\mu$$

and, similarly

$$\int_{A^c} g\,\mathrm{d}\mu \leqslant \int_{A^c} Sg\,\mathrm{d}\mu,$$

which implies $\mu_i \in \mathrm{P}_{\mathcal{S}}(K)$ for $i = 1,2$. Moreover,

$$\mu = \mu(A)\mu_1 + (1-\mu(A))\mu_2,$$

so $\mu \notin \mathrm{ex}\,\mathrm{P}_{\mathcal{S}}(K)$.

Conversely, take an ergodic measure $\mu \in \mathrm{P}_{\mathcal{S}}(K)$ and suppose that $\mu = \frac{1}{2}(\mu_1 + \mu_2)$ for some $\mu_1, \mu_2 \in \mathrm{P}_{\mathcal{S}}(K)$. Since

$$|\langle f, \mu_1\rangle| \leqslant 2\langle |f|, \mu\rangle \leqslant 2\|f\|_{\mathrm{L}^1(K,\mu)}$$

for each $f \in \mathrm{C}(K)$ and $\mathrm{C}(K)$ is dense in $\mathrm{L}^1(K,\mu)$, we conclude that μ_1 extends uniquely to a continuous functional $\tilde{\mu}_1 \in \mathrm{L}^\infty(K,\mu) = \mathrm{L}^1(K,\mu)'$. The semigroup $\mathcal{S}_\mu$ is mean ergodic (in the sense of Definition 8.31 of [EFHN15]) on $\mathrm{L}^1(K,\mu)$ (see Example 13.24 of [EFHN15]) and therefore $\mathrm{fix}(\mathcal{S}_\mu)$ separates $\mathrm{fix}(\mathcal{S}_\mu')$ by Theorem 8.33 of [EFHN15]. Since $\mathrm{fix}(\mathcal{S}_\mu)$ is one-dimensional,

$\mathrm{fix}(\mathcal{S}'_\mu)$ is also one-dimensional. Consequently we obtain $\tilde{\mu}_1 = \mathbb{1} \in \mathrm{L}^\infty(K,\mu)$ which implies $\mu_1 = \mu$. □

We are now ready to introduce primitive $\mathcal{S}$-ideals.

Definition 2.6. An $\mathcal{S}$-ideal p is called *primitive* if there is an ergodic measure $\mu \in \mathrm{P}_\mathcal{S}(K)$ with $p = I_\mu$.
The set of all primitive $\mathcal{S}$-ideals is called the *primitive spectrum of* $\mathcal{S}$, denoted by $\mathrm{Prim}(\mathcal{S})$.

Remark 2.7. The supports of primitive $\mathcal{S}$-ideals are precisely the supports of ergodic measures. Instead of looking at the ideal space it is therefore justified (and sometimes helpful) to see the primitive spectrum as a subset of the power set of K.

Remark 2.8. If $\mathcal{S} \subseteq \mathscr{L}(\mathrm{C}(K))$ is irreducible, i.e., there are no non-trivial $\mathcal{S}$-ideals, then $\mathrm{Prim}(\mathcal{S})$ is a singleton. Other examples are given below (cf. Examples 5.7).

We need the following result which relates invariant measures for quotient systems to invariant measures on K.

Proposition 2.9. *Let $L \subseteq K$ be the support of an $\mathcal{S}$-ideal and consider the semigroup $\mathcal{S}_L$ of Markov operators on $\mathrm{C}(L)$ induced by $\mathcal{S}$. The canonical continuous embedding*

$$i\colon \mathrm{C}(L)' \longrightarrow \mathrm{C}(K)'$$

with $i(\mu)(f) := \langle f|_L, \mu\rangle$ for each $f \in \mathrm{C}(K)$ and $\mu \in \mathrm{C}(L)'$ restricts to continuous embeddings

$$\begin{aligned} i\colon \mathrm{P}_{\mathcal{S}_L}(L) &\longrightarrow \mathrm{P}_\mathcal{S}(K),\\ i\colon \mathrm{ex}\,\mathrm{P}_{\mathcal{S}_L}(L) &\longrightarrow \mathrm{ex}\,\mathrm{P}_\mathcal{S}(K) \end{aligned}$$

with

$$\begin{aligned} i(\mathrm{P}_{\mathcal{S}_L}(L)) &= \{\tilde{\mu} \in \mathrm{P}_\mathcal{S}(K) \mid \mathrm{supp}\,\tilde{\mu} \subseteq L\} \text{ and}\\ i(\mathrm{ex}\,\mathrm{P}_{\mathcal{S}_L}(L)) &= \{\tilde{\mu} \in \mathrm{ex}\,\mathrm{P}_\mathcal{S}(K) \mid \mathrm{supp}\,\tilde{\mu} \subseteq L\}. \end{aligned}$$

Proof. It is obvious that images of invariant measures remain invariant. Now assume that $\mu \in \mathrm{ex}\,\mathrm{P}_\mathcal{S}(L)$ and suppose that $i(\mu) = \frac{1}{2}(\tilde{\mu}_1 + \tilde{\mu}_2)$ for measures $\tilde{\mu}_1, \tilde{\mu}_2 \in \mathrm{P}_\mathcal{S}(K)$. Then $\mathrm{supp}\,\tilde{\mu}_i \subseteq \mathrm{supp}\,\mu$ for $i = 1, 2$ and therefore $\tilde{\mu}_1$ and $\tilde{\mu}_2$ restrict to measures $\mu_1, \mu_2 \in \mathrm{P}_{\mathcal{S}_L}(L)$ with $\mu = \frac{1}{2}(\mu_1 + \mu_2)$, so $\mu_1 = \mu = \mu_2$ since μ is ergodic. □

Corollary 2.10. *Each maximal $\mathcal{S}$-ideal is primitive.*

Proof. Take a maximal $\mathcal{S}$-ideal $I = I_L$. Then the induced semigroup $\mathcal{S}_I$ on $\mathrm{C}(L)$ is irreducible and consequently every S_I-invariant measure μ is strictly positive, i.e., $\mathrm{supp}\,\mu = L$.

□

We give two simple examples showing that the converse of Corollary 2.10 does not hold.

Example 2.11.
(i) If $K = \mathbb{T} := \{z \in \mathbb{C} \mid |z| = 1\}$ and $\varphi(z) := z^2$ for $z \in \mathbb{T}$, then the Haar measure of $\mathbb{T}$ is ergodic by Proposition 2.17 of [EW11]. However, $1 \in \mathbb{T}$ is a fixed point of φ and therefore $I_{\{1\}}$ is a non-trivial $\mathcal{S}_\varphi$-ideal. Therefore, the zero ideal is a primitive, but not maximal $\mathcal{S}_\varphi$-ideal.
(ii) Consider $K = \{0,1\}^{\mathbb{N}}$ and $\varphi((x_n)_{n\in\mathbb{N}}) := (x_{n+1})_{n\in\mathbb{N}}$ for $(x_n)_{n\in\mathbb{N}} \in K$. Clearly, φ has fixed points whence the zero ideal is not a maximal $\mathcal{S}_\varphi$-ideal.
Let $\nu := \frac{1}{2}(\delta_0 + \delta_1) \in \mathrm{C}(\{0,1\})'$. Then the product measure $\mu := \prod_{n\in\mathbb{N}} \nu \in \mathrm{C}(K)'$ on K is ergodic by Proposition 6.20 of [EFHN15] and has full support. Therefore $I_\mu = \{0\}$ is a primitive $\mathcal{S}_\varphi$-ideal.

Remark 2.12. In view of Examples 2.11 considering all primitive ideals instead of maximal ideals yields more information on the semigroup action.

3. Radical Ideals

The Jacobson topology on the primitive spectrum of C*-algebras can be defined using the notions of *hull* and *kernel* (see Section 4.3 of [Ped79] or Section II.6.5 of [Bla06]). In our context they also yield a natural correspondence between closed subsets of $\mathrm{Prim}(\mathcal{S})$ and so-called *radical $\mathcal{S}$-ideals*.

Definition 3.1. For subsets $A \subseteq \mathrm{Prim}(\mathcal{S})$ and $I \subseteq \mathrm{C}(K)$ we set

$$\mathrm{ker}(A) := \bigcap_{p\in A} p,$$
$$\mathrm{hull}(I) := \left\{p \in \mathrm{Prim}(\mathcal{S}) \mid I \subseteq p\right\}.$$

(i) For a subset $I \subseteq \mathrm{C}(K)$ the $\mathcal{S}$-*radical* of I is

$$\mathrm{rad}_{\mathcal{S}}(I) := \mathrm{ker}(\mathrm{hull}(I))) = \bigcap_{\substack{p\in\mathrm{Prim}(\mathcal{S})\\ I\subseteq p}} p.$$

(ii) An $\mathcal{S}$-ideal I is a *radical $\mathcal{S}$-ideal* if $I = \mathrm{rad}_{\mathcal{S}}(I)$.
(iii) The semigroup $\mathcal{S}$ is *radical free* if the zero ideal is a radical $\mathcal{S}$-ideal, i.e., if $\mathrm{rad}_{\mathcal{S}}(0) = 0$.

We denote the set of all radical $\mathcal{S}$-ideals by $\mathrm{Rad}(\mathcal{S})$.

Remark 3.2. We point out that our definition of a radical free semigroup does not coincide with the one of Schaefer (using maximal $\mathcal{S}$-ideals, see [Sch68]). Every radical free semigroup in the sense of Schaefer is also radical free in our terminology. However, the converse does not hold (see Examples 2.11).

Remark 3.3. By the Krein-Milman theorem $\mathrm{P}_{\mathcal{S}}(K)$ is the closed convex hull of $\mathrm{ex}\,\mathrm{P}_{\mathcal{S}}(K)$ with respect to the weak* topology and therefore

$$\mathrm{rad}_{\mathcal{S}}(0) = \bigcap_{\mu\in\mathrm{P}_{\mathcal{S}}(K)} I_\mu.$$

Remark 3.4. The $\mathcal{S}$-radical of a subset $I \subseteq \mathrm{C}(K)$ is either $\mathrm{C}(K)$ or a radical $\mathcal{S}$-ideal. Moreover, we always have $\mathrm{hull}(I) = \mathrm{hull}(\mathrm{rad}_{\mathcal{S}}(I))$.

Remark 3.5. Just as primitive ideals correspond to the supports of ergodic measures, radical ideals correspond to the closures of unions of supports of ergodic measures. Therefore $\mathcal{S}$ is radical free if and only if the union of all supports of invariant ergodic measures is dense in K. Note that the latter set is not closed in general (see Example 5.7 (iii) below). For the Markov semigroup induced by the right shift on $K = \beta\mathbb{N}\backslash\mathbb{N}$ this set is nowhere dense (cf. Corollary 1.5 in [Cho67]).

We need the following result which relates radical and primitive ideals of quotient systems to the corresponding $\mathcal{S}$-ideals of $\mathrm{C}(K)$.

Proposition 3.6. *Let $I = I_L \subseteq \mathrm{C}(K)$ be an $\mathcal{S}$-ideal and $\mathcal{S}_I$ the semigroup of Markov operators on $\mathrm{C}(L)$ induced by $\mathcal{S}$. Then the mappings*

$$\begin{aligned} \{p \in \mathrm{Prim}(\mathcal{S}) \mid I \subseteq p\} &\longrightarrow \mathrm{Prim}(\mathcal{S}_I), \quad p \mapsto p|_L, \\ \{J \in \mathrm{Rad}(\mathcal{S}) \mid I \subseteq J\} &\longrightarrow \mathrm{Rad}(\mathcal{S}_I), \quad J \mapsto J|_L, \end{aligned}$$

where $J|_L := \{f|_L \mid f \in J\}$ for $J \subseteq \mathrm{C}(K)$, are inclusion preserving bijections with inclusion preserving inverses. Moreover, $\mathrm{rad}_{\mathcal{S}_I}(0) = \mathrm{rad}_{\mathcal{S}}(I)|_L$.

Proof. We first recall that the natural projection $P\colon \mathrm{C}(K) \longrightarrow \mathrm{C}(L)$ is a surjetive Banach lattice homomorphism. Thus, if $\tilde{J} \subseteq \mathrm{C}(L)$ is a closed ideal, then $J := P^{-1}(\tilde{J})$ is a closed ideal containing I with $\tilde{J} = P(P^{-1}(\tilde{J})) = J|_L$. It is readily checked that J is the unique closed ideal H containing I with $H|_L = \tilde{J}$. Clearly $\tilde{J}$ is $\mathcal{S}_I$-invariant if and only if J is $\mathcal{S}$-invariant.
We therefore obtain mutually inverse and inclusion preserving mappings

$$\begin{aligned} \{J \subseteq \mathrm{C}(K) \mid J\, \mathcal{S}\text{-ideal with } I \subseteq J\} &\leftrightarrow \{\tilde{J} \subseteq \mathrm{C}(L) \mid \tilde{J}\, \mathcal{S}_I\text{-ideal}\}, \\ J &\mapsto J|_L \\ P^{-1}(\tilde{J}) &\leftarrow\!\shortmid \tilde{J}. \end{aligned}$$

We now prove that

$$\{p \in \mathrm{Prim}(\mathcal{S}) \mid I \subseteq p\} \longrightarrow \mathrm{Prim}(\mathcal{S}_I), \quad p \mapsto p|_L$$

is a bijective map. Assume that $I \subseteq J = I_\mu$ for some $\mu \in \mathrm{ex}\, \mathrm{P}_{\mathcal{S}}(K)$. Then $\mathrm{supp}\, \mu \subseteq L$ and we thus find $\nu \in \mathrm{ex}\, \mathrm{P}_{\mathcal{S}_I}(L)$ with $i(\nu) = \mu$ (see Proposition 2.9). Moreover we obtain for every $f \in \mathrm{C}(L)$ that

$$\langle |f|, \nu \rangle = \int_L |F| \,\mathrm{d}\mu, \tag{1}$$

for each extension $F \in \mathrm{C}(K)$ of f to K. Thus $f \in I_\nu$ if and only if $f \in I_\mu|_L$. If, on the other hand, $\tilde{J} = I_\nu$ for some $\nu \in \mathrm{ex}\, \mathrm{P}_{\mathcal{S}_I}(L)$, then Equation (1) holds for $\mu = i(\nu)$ and thus $\tilde{J} = I_\mu|_L$.
Before proceeding with the remaining assertions, we make the following two observations.

- For a familiy $(J_\alpha)_{\alpha\in A}$ of $\mathcal{S}$-ideals with $I \subseteq J_\alpha$ for every $\alpha \in A$
$$\left(\bigcap_{\alpha\in A} J_\alpha\right)|_L = \bigcap_{\alpha\in A}(J_\alpha|_L).$$
- For two $\mathcal{S}$-ideals J_1, J_2 with $I \subseteq J_1, J_2$ the inclusion $J_1|_L \subseteq J_2|_L$ implies $J_1 \subseteq J_2$.

We use these facts to show that
$$\{J \in \mathrm{Rad}(\mathcal{S}) \mid I \subseteq J\} \longrightarrow \mathrm{Rad}(\mathcal{S}_I), \quad J \mapsto J|_L$$
is a well-defined bijection. Take an $\mathcal{S}$-ideal $J \subseteq \mathrm{C}(K)$ with $I \subseteq J$. Then J is radical if and only if
$$J = \bigcap_{\substack{p\in\mathrm{Prim}(\mathcal{S})\\ J\subseteq p}} p$$
which is—by the observations above—equivalent to
$$J|_L = \bigcap_{\substack{p\in\mathrm{Prim}(\mathcal{S})\\ J\subseteq p}} p|_L = \bigcap_{\substack{p\in\mathrm{Prim}(\mathcal{S})\\ J|_L\subseteq p|_L}} p|_L = \bigcap_{\substack{p\in\mathrm{Prim}(\mathcal{S}_I)\\ J|_L\subseteq p}} p,$$
i.e., $J|_L$ being a radical $\mathcal{S}_I$-ideal.
Finally, the identity $\mathrm{rad}_{\mathcal{S}_I}(0) = \mathrm{rad}_{\mathcal{S}}(I)|_L$ follows from the fact that $\mathrm{rad}_{\mathcal{S}_I}(0)$ is the smallest radical $\mathcal{S}_I$-ideal and $\mathrm{rad}_{\mathcal{S}}(I)$ is the smallest radical $\mathcal{S}$-ideal containing I. □

Our main class of examples for radical ideals are the absolute kernels of (possibly non-ergodic) invariant measures. The following result generalizes Proposition 12 of [Sch68] using similar arguments.

Proposition 3.7. *The following assertions are valid.*

(i) *For each $\mu \in \mathrm{P}_{\mathcal{S}}(K)$ the $\mathcal{S}$-ideal I_μ is a radical $\mathcal{S}$-ideal.*

(ii) *If K is metrizable, then for each radical $\mathcal{S}$-ideal I there is $\mu \in \mathrm{P}_{\mathcal{S}}(K)$ with $I = I_\mu$.*

Proof. For (i) let $\mu \in \mathrm{P}_{\mathcal{S}}(K)$. By Lemma 3.6 we may assume that $K = \operatorname{supp}\mu$ and it then suffices to show that $\mathcal{S}$ is radical free. But this directly follows from Remark 3.3.

We now prove (ii) and assume that K is metrizable and I is a radical $\mathcal{S}$-ideal. We may assume that $I = 0$ (otherwise we pass to $\mathrm{C}(\operatorname{supp} I)$, cf. Lemma 3.6). Take a countable base of the topology consisting of nonempty open sets U_n, $n \in \mathbb{N}$. Since the supports of ergodic measures are dense in K we find $\mu_n \in \operatorname{ex}\mathrm{P}_{\mathcal{S}}(K)$ with $\operatorname{supp}\mu_n \cap U_n \neq \emptyset$ for each $n \in \mathbb{N}$. For
$$\mu := \sum_{n=1}^\infty 2^{-n}\mu_n \in \mathrm{P}_{\mathcal{S}}(K)$$
we obtain $\mu(U_n) > 0$ for each $n \in \mathbb{N}$, hence $\mu(U) > 0$ for each non-empty open set $U \subseteq K$. □

Remark 3.8. Taking $\mathcal{S} = \{\mathrm{Id}\}$ in Proposition 3.7 (ii) yields the probably well-known fact that every compact metric space has a fully supported regular Borel probability measure.[1]

The following examples show that part (ii) of Proposition 3.7 is wrong in the non-metric case.

Example 3.9. (i) If $K = \Omega \cup \{\infty\}$ is the one-point compactification of an uncountable discrete space Ω, then $\mathrm{C}(K)'$ can be identified with $\ell^1(K)$. Thus there is no fully supported probability measure $\mu \in \mathrm{C}(K)'$.[1]

(ii) If $K = \beta\mathbb{N} \setminus \mathbb{N}$ and $\mathcal{S}_\varphi$ is the Markov semigroup induced by the right shift φ, then

$$\bigcap_{n\in\mathbb{N}} I_{\mu_n} \not\subseteq \mathrm{rad}_{\mathcal{S}_\varphi}(0)$$

for every sequence of probability measures $(\mu_n)_{n\in\mathbb{N}} \subseteq \mathrm{C}(K)'$ (see Corollary 1.10 of [Cho67]).

4. Centers of Attraction

Radical $\mathcal{S}$-ideals can also be described via an ergodic stability condition. To formulate our theorem we write $\overline{\mathrm{co}}\,\mathcal{S}$ for the closed convex hull of $\mathcal{S}$ with respect to the strong operator topology and recall that a net $(T_\alpha)_{\alpha\in A} \subseteq \overline{\mathrm{co}}\,\mathcal{S} \subseteq \mathscr{L}(\mathrm{C}(K))$ of operators is *right ergodic* if

$$\lim_\alpha T_\alpha(\mathrm{Id} - S) = 0$$

for each $S \in \mathcal{S}$ with respect to the strong operator topology. We note that there always are right ergodic operator nets for $\mathcal{S}$ (see Corollary 1.5 of [Sch13]). We give some examples (see Examples 1.2 of [Sch13]).

Example 4.1. (i) If $\mathcal{S} = \{S^n \mid n \in \mathbb{N}_0\}$ for some Markov operator $S \in \mathscr{L}(\mathrm{C}(K))$, then the *Cesàro means*

$$C_N := \frac{1}{N}\sum_{n=0}^{N-1} S^n \text{ for } N \in \mathbb{N}$$

define a right ergodic operator sequence $(C_N)_{N\in\mathbb{N}}$ for $\mathcal{S}$. Likewise, the net of *Abel means* $(A_r)_{r\in(0,1)}$ defined by

$$A_r := (1-r)\sum_{n=0}^{\infty}(rS)^n \text{ for } r \in (0,1)$$

is right ergodic for $\mathcal{S}$.

(ii) If $\mathcal{S} = \{S(t) \mid t \geqslant 0\}$ is a strongly continuous one-parameter semigroup of Markov operators on $\mathrm{C}(K)$, then the *Cesàro means*

$$C_T f := \frac{1}{T}\int_0^T S(t)f\,\mathrm{d}t \text{ for } f \in \mathrm{C}(K) \text{ and } T > 0$$

define a right ergodic operator net $(C_T)_{T>0}$ for $\mathcal{S}$.

[1]Remark 3.8 and Example 3.9 (i) were kindly suggested by the referee.

The following result generalizes Theorem 4 of [Sch68].

Theorem 4.2. *For each support $L \subseteq K$ of an $\mathcal{S}$-ideal*

$$\begin{aligned}\mathrm{rad}_{\mathcal{S}}(I_L) &= \left\{ f \in \mathrm{C}(K) \,\middle|\, \lim_{\alpha} \int_L T_\alpha |f| \,\mathrm{d}\mu = 0 \textit{ for each } \mu \in \mathrm{C}(L)' \right\} \\ &= \left\{ f \in \mathrm{C}(K) \,\middle|\, \lim_{\alpha} (T_\alpha |f|)|_L = 0 \textit{ in the norm of } \mathrm{C}(L) \right\}\end{aligned}$$

where $(T_\alpha)_{\alpha \in A}$ is any right ergodic operator net for $\mathcal{S}$.
In particular, if $(T_\alpha)_{\alpha \in A}$ is any right ergodic operator net for $\mathcal{S}$, then an $\mathcal{S}$-ideal I_L is a radical $\mathcal{S}$-ideal if and only if every $f \in \mathrm{C}(K)$ satisfying

$$\lim_{\alpha} (T_\alpha |f|)|_L = 0$$

vanishes on L.

Proof. By Lemma 3.6 we may assume $L = K$. Take $f \in \mathrm{rad}_{\mathcal{S}}(0)$ and any right ergodic operator net $(T_\alpha)_{\alpha \in A}$ for $\mathcal{S}$.
Let $\mu \in \mathrm{C}(K)'$ and observe that each subnet of $(T'_\alpha \mu)_{\alpha \in A}$ has a subnet converging to some $\nu \in \mathrm{P}_{\mathcal{S}}(K)$. Since $\langle |f|, \nu \rangle = 0$ (see Remark 3.3), we obtain that each subnet of $(\langle T_\alpha |f|, \mu \rangle)_{\alpha \in A}$ has a subnet converging to zero which implies

$$\lim_{\alpha} \langle T_\alpha |f|, \mu \rangle = 0.$$

Now let $f \in \mathrm{C}(K)$ with $\lim_\alpha T_\alpha |f| = 0$ weakly for some right ergodic operator net $(T_\alpha)_{\alpha \in A}$ for $\mathcal{S}$. Then for $\mu \in \mathrm{P}_{\mathcal{S}}(K)$

$$0 = \lim_{\alpha} \langle T_\alpha |f|, \mu \rangle = \langle |f|, \mu \rangle$$

which proves $f \in \mathrm{rad}_{\mathcal{S}}(0)$ and thus the first equation.
By Theorem 1.7 of [Sch13] the semigroup $\mathcal{S}$ is mean ergodic on $\mathrm{rad}_{\mathcal{S}}(0)$ with mean ergodic projection $P = 0$ and therefore

$$\mathrm{rad}_{\mathcal{S}}(0) \subseteq \left\{ f \in \mathrm{C}(K) \,\middle|\, \lim_{\alpha} T_\alpha |f| = 0 \text{ in the norm of } \mathrm{C}(K) \right\}.$$

The converse inclusion is obvious. □

If $\mathcal{S}$ has a right ergodic operator sequence (for example if it has a Følner sequence as defined in Assumption 4.5 below), then Lebesgue's Theorem yields the following result.

Corollary 4.3. *Suppose that $(T_n)_{n \in \mathbb{N}}$ is a right ergodic operator sequence for $\mathcal{S}$. Then*

$$\mathrm{rad}_{\mathcal{S}}(I) = \left\{ f \in \mathrm{C}(K) \,\middle|\, \lim_{n \to \infty} T_n |f|(x) = 0 \textit{ for each } x \in \operatorname{supp} I \right\}$$

for each $\mathcal{S}$-ideal I.

The next corollary shows that if $\mathcal{S}$ is the semigroup generated by a Markov lattice homomorphism $T \in \mathscr{L}(\mathrm{C}(K))$ (i.e., a Koopman operator), the radical $\mathrm{rad}_{\mathcal{S}}(0)$ of the zero ideal coincides with the almost weakly stable part of $\mathrm{C}(K)$ with respect to T as defined in (9.4) on page 176 of [EFHN15].

Corollary 4.4. *Assume that* $\varphi\colon K \longrightarrow K$ *is a continuous mapping and* $\mathcal{S} = \mathcal{S}_\varphi$. *Then*

$$\mathrm{rad}_{\mathcal{S}}(0) = \left\{ f \in \mathrm{C}(K) \,\middle|\, \lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} |\langle T_\varphi^n f, \mu\rangle| = 0 \text{ for each } \mu \in \mathrm{C}(K)' \right\}.$$

Proof. If $f \in \mathrm{rad}_{\mathcal{S}}(0)$ and $\mu \in \mathrm{C}(K)'$, we obtain

$$\frac{1}{N} \sum_{n=0}^{N-1} |\langle T_\varphi^n f, \mu\rangle| \leqslant \frac{1}{N} \sum_{n=0}^{N-1} \langle T_\varphi^n |f|, |\mu|\rangle$$

for every $N \in \mathbb{N}$ and therefore $\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} |\langle T_\varphi^n f, \mu\rangle| = 0$ by Theorem 4.2. The converse inclusion follows directly from Corollary 4.3. $\square$

For Koopman semigroups $\mathcal{S}$ we also obtain a further dynamical characterization of $\mathrm{rad}_{\mathcal{S}}(0)$. For the rest of this section we make the following assumption (cf. Examples 1.2 (e) of [Sch13]).

Assumption 4.5. Let $\mathcal{S}$ be a closed subsemigroup of a locally compact group $\mathcal{G}$ with left-invariant Haar measure λ acting on K such that

$$\mathcal{S} \times K \longrightarrow K, \quad (s, x) \mapsto sx$$

is continuous. Let $\mathcal{S}$ be the associated Koopman semigroup, i.e.,

$$\mathcal{S} = \{T_s \mid s \in \mathcal{S}\}$$

with $T_s f(x) := f(sx)$ for $f \in \mathrm{C}(K)$, $s \in \mathcal{S}$ and $x \in K$, which is strongly continuous by Theorem 4.17 of [EFHN15]. Moreover, we assume that $(F_n)_{n\in\mathbb{N}}$ is a *Følner sequence* for $\mathcal{S}$, i.e., each F_n is a compact subset of $\mathcal{S}$ with positive measure satisfying

$$\lim_{n\to\infty} \frac{\lambda(F_n \Delta s F_n)}{\lambda(F_n)} = 0$$

for each $s \in \mathcal{S}$.

Example 4.6. (i) If $\mathcal{S}$ is the additive semigroup $\mathbb{N}_0$, then the sequence $(F_n)_{n\in\mathbb{N}}$ defined by $F_n := \{0, ..., n-1\}$ for $n \in \mathbb{N}$ is a Følner sequence for $\mathcal{S}$.

(ii) If $\mathcal{S}$ is the additive semigroup $\mathbb{R}_{\geqslant 0}$ and $(t_n)_{n\in\mathbb{N}}$ is any sequence in $(0, \infty)$ with $\lim_{n\to\infty} t_n = \infty$ then $(F_n)_{n\in\mathbb{N}}$ defined by $F_n := [0, t_n]$ for $n \in \mathbb{N}$ is a Følner sequence for $\mathcal{S}$.

Lemma 4.7. *Under Assumption 4.5* $\mathcal{S}$ *is left amenable and thus* $\mathcal{S}$ *is right amenable. Moreover we obtain an ergodic operator sequence* $(\mathcal{F}_n)_{n\in\mathbb{N}}$ *for* $\mathcal{S}$ *by setting*

$$\mathcal{F}_n f := \frac{1}{\lambda(F_n)} \int_{F_n} T_s f \,\mathrm{d}\lambda(s)$$

for $f \in \mathrm{C}(K)$ *and* $n \in \mathbb{N}$.

Proof. For each $n \in \mathbb{N}$ set

$$m_n(f) := \frac{1}{\lambda(F_n)} \int_{F_n} f(s)\, \mathrm{d}\lambda(s)$$

for $f \in \mathrm{C_b}(\mathcal{S})$. Then $m_n \in \mathrm{C_b}(\mathcal{S})'$ with $m_n(\mathbb{1}) = 1$ and $m_n \geqslant 0$ for each $n \in \mathbb{N}$. Let m be any weak* limit point of $(m_n)_{n\in\mathbb{N}}$. Since

$$\left| \frac{1}{\lambda(F_n)} \int_{F_n} f(s)\, \mathrm{d}\lambda(s) - \frac{1}{\lambda(F_n)} \int_{F_n} f(ts)\, \mathrm{d}\lambda(s) \right| \leqslant \frac{\lambda(F_n \Delta t F_n)}{\lambda(F_n)} \cdot \|f\|$$

for each $f \in \mathrm{C_b}(\mathcal{S})$, $n \in \mathbb{N}$ and $t \in \mathcal{S}$, m is an invariant mean. The second assertion is obvious. □

We now introduce certain "attractors" of the dynamical system $(K; S)$ with respect to the Følner sequence $(F_n)_{n\in\mathbb{N}}$.

Definition 4.8. A closed non-empty set $L \subseteq K$ is a *(global) center of attraction* if for each open set $U \supseteq L$ we have

$$\lim_{n\to\infty} \frac{1}{\lambda(F_n)} \lambda(\{s \in F_n \mid sx \in U\}) = 1$$

for every $x \in K$.

This type of attraction is quite weak. Loosely speaking, orbits of points may move arbitarily far away from a center of attraction as long as they come back "often enough" with respect to the Følner sequence.
Global as well as point-dependent centers of attraction for $\mathbb{N}_0$- and $\mathbb{R}_{\geqslant 0}$-actions have been examined by several authors (see, e.g., [Hil36], [Ber51], [JR72], [Sig77], Exercise I.8.3 in [Man87] and [Dai16]). In a recent paper Z. Chen and X. Dai study the chaotic behavior of minimal centers of attraction with respect to a point for discrete amenable group actions (see [CD17]).
It is known that in case of $\mathbb{N}_0$-actions on metric compact spaces there always is a unique minimal (global) center of attraction given by the closure of the union of the supports of ergodic measures (see Exercises I.8.3 and II.1.5 in [Man87]). The following result shows that this still holds in our more general situation.

Theorem 4.9. *Under Assumption 4.5 the definition of a center of attraction does not depend on the Følner sequence. Moreover, for a closed non-empty set $L \subseteq K$ the following assertions are equivalent.*

(a) *L is a center of attraction.*

(b) *$I_L := \{f \in \mathrm{C}(K) \mid f|_L = 0\} \subseteq \mathrm{rad}_{\mathcal{S}}(0)$.*

In particular there is a unique minimal center of attraction $\mathrm{M}(\mathcal{S})$ *given by the closure of the union of the supports of ergodic measures, i.e.,*

$$\mathrm{M}(\mathcal{S}) = \operatorname{supp} \mathrm{rad}_{\mathcal{S}}(0).$$

Proof. Take a non-empty and closed set $L \subseteq K$. The mapping

$$I_L \longrightarrow \mathrm{C}_0(K \setminus L), \quad f \mapsto f|_{K\setminus L}$$

is an isomorphism of Banach lattices. Now L is a center of attraction if and only if

$$\lim_{n\to\infty} \frac{1}{\lambda(F_n)} \lambda(\{s \in F_n \mid sx \in A\}) = 0,$$

i.e.,

$$\lim_{n\to\infty} \frac{1}{\lambda(F_n)} \int_{F_n} \mathbb{1}_A(sx)\,\mathrm{d}\lambda(s) = 0 \tag{2}$$

for each compact set $A \subseteq K \setminus L$ and each $x \in K$. Since the mapping

$$\mathcal{S} \times K \longrightarrow K, \quad (s,x) \mapsto sx$$

is continuous, the function $f\colon \mathcal{S}\times K \longrightarrow \mathbb{C}$, $(s,x) \mapsto \mathbb{1}_A(sx)$ is Borel measurable. By Lebesgue's and Fubini's theorems Equation (2) is thus equivalent to

$$\lim_{n\to\infty} \int_K \int_{F_n} \mathbb{1}_A(sx)\,\mathrm{d}\lambda(s)\,\mathrm{d}\mu(x) = \lim_{n\to\infty} \frac{1}{\lambda(F_n)} \int_{F_n} T_s'\mu(A)\,\mathrm{d}\lambda(s) = 0$$

for each $\mu \in \mathrm{C}_0(K \setminus L)'$ and each compact set $A \subseteq K \setminus L$. Since the space of compactly supported continuous functions $\mathrm{C}_\mathrm{c}(K \setminus L)$ is dense in $\mathrm{C}_0(K \setminus L)$, this is the case if and only if

$$\lim_{n\to\infty} \frac{1}{\lambda(F_n)} \int_{F_n} \langle T_s|f|, \mu\rangle \mathrm{d}\lambda(s) = 0$$

for each $f \in I_L$ and every $\mu \in I_L'$. This means

$$\lim_{n\to\infty} \mathcal{F}_n|f| = 0$$

with respect to the weak topology for each $f \in I_L$, i.e., $I_L \subseteq \mathrm{rad}_\mathcal{S}(0)$. □

5. The Primitive Spectrum as a Topological Space

In this section we return to a general right amenable Markov semigroup $\mathcal{S} \subseteq \mathscr{L}(\mathrm{C}(K))$ and analyze the topology of $\mathrm{Prim}(\mathcal{S})$. It turns out that it basically has the same properties as the (non-dynamical) primitive spectrum of C*-algebras and the topology of affine schemes of algebraic geometry (cf. Section (2.2) of [GW10]). We employ methods as in Chapter IV of [Dix77] and Section 4.3 of [Ped79] and first prove two technical lemmas before introducing a topology on $\mathrm{Prim}(\mathcal{S})$. Recall that given $\mu \in \mathrm{P}_\mathcal{S}(K)$ we write $\mathcal{S}_\mu$ for the induced semigroup on $\mathrm{L}^1(K,\mu)$.

Lemma 5.1. *If $\mu \in \mathrm{P}_\mathcal{S}(K)$ and $L \subseteq K$ is the support of an $\mathcal{S}$-ideal, then $\mathbb{1}_L \in \mathrm{fix}(\mathcal{S}_\mu)$.*

Proof. We fix $S \in \mathcal{S}$. For each open set $O \supseteq L$ take a continuous function f_O with $f_O(K) \subseteq [0,1]$, $f|_L = \mathbb{1}$ and $f_O|_{(K\setminus O)} = 0$. The set Λ of open sets containing L is directed with respect to converse set inclusion and thus we obtain a net $(f_O)_{O\in\Lambda}$ with

$$\|\mathbb{1}_L - f_O\|_{\mathrm{L}^1(K,\mu)} \leqslant \mu(O \setminus L) \to 0$$

by regularity of μ. By definition of $\mathcal{S}_\mu$

$$S_\mu \mathbb{1}_L = \lim_O S f_O.$$

in $\mathrm{L}^1(K,\mu)$. Moreover,

$$Sf_O(x) = \langle Sf_O, \delta_x\rangle = \langle f_O, S'\delta_x\rangle = 1$$

for each $x \in L$ since L is the support of an $\mathcal{S}$-ideal. This implies

$$0 = \lim_O ((\mathbb{1} - Sf_O)\cdot \mathbb{1}_L) = \mathbb{1}_L - S_\mu \mathbb{1}_L \cdot \mathbb{1}_L,$$

where the limit is taken in $\mathrm{L}^1(K,\mu)$. Thus $\mathbb{1}_L = S_\mu \mathbb{1}_L \cdot \mathbb{1}_L$ which shows $\mathbb{1}_L \leqslant S_\mu \mathbb{1}_L$ and consequently $\mathbb{1}_L \in \mathrm{fix}(\mathcal{S}_\mu)$ by Theorem 13.2 (d) of [EFHN15]. □

Lemma 5.2. *Consider two $\mathcal{S}$-ideals I_1, I_2. If p is a primitive $\mathcal{S}$-ideal with $I_1 \cap I_2 \subseteq p$, then $I_1 \subseteq p$ or $I_2 \subseteq p$.*

Proof. Let $p = I_\mu$ for some $\mu \in \mathrm{ex}\,\mathrm{P}_{\mathcal{S}}(K)$ and let $L_j := \mathrm{supp}\, I_j$ for $j = 1,2$. By Lemma 5.1, $\mathbb{1}_{L_j} \in \mathrm{fix}(\mathcal{S}_\mu)$ for $j = 1,2$ and therefore $\mu(L_j) \in \{0,1\}$ since μ is ergodic. Now $\mathrm{supp}\,\mu \subseteq L_1 \cup L_2$ implies $\mu(L_1 \cup L_2) = 1$, so there is $j \in \{1,2\}$ with $\mu(L_j) = 1$. But this means $\mathrm{supp}\,\mu \subseteq L_j$ and consequently $I_j \subseteq p$. □

We are now ready to equip $\mathrm{Prim}(\mathcal{S})$ with a topology by defining a Kuratowski closure operator (see page 43 of [Kel75] for this notion). Recall that the concepts of *hull* and *kernel* have been introduced in Definition 3.1.

Proposition 5.3. *The mapping*

$$\overline{}\colon \mathscr{P}(\mathrm{Prim}(\mathcal{S})) \longrightarrow \mathscr{P}(\mathrm{Prim}(\mathcal{S})), \quad A \mapsto \overline{A} := \mathrm{hull}(\ker(A))$$

defines a Kuratowski closure operator.

Proof. It is readily checked that

$$\overline{\emptyset} = \emptyset,\ A \subseteq \overline{A} \text{ and } \overline{\overline{A}} = \overline{A}$$

for each $A \subseteq \mathrm{Prim}(\mathcal{S})$. It remains to show that $\overline{A_1 \cup A_2} = \overline{A_1} \cup \overline{A_2}$ for all $A_1, A_2 \subseteq \mathrm{Prim}(\mathcal{S})$. Applying Lemma 5.2 to the ideals $I_j := \ker(A_j)$ for $j = 1,2$ yields

$$\begin{aligned}\overline{A_1 \cup A_2} &= \mathrm{hull}(\ker(A_1 \cup A_2)) = \mathrm{hull}(\ker(A_1) \cap \ker(A_2))\\ &= \{p \in \mathrm{Prim}(\mathcal{S}) \mid \ker(A_1) \cap \ker(A_2) \subseteq p\}\\ &= \{p \in \mathrm{Prim}(\mathcal{S}) \mid \ker(A_1) \subseteq p \text{ or } \ker(A_2) \subseteq p\} = \overline{A_1} \cup \overline{A_2},\end{aligned}$$

which shows the claim. □

Definition 5.4. The topology on $\mathrm{Prim}(\mathcal{S})$ induced by the closure operator of Proposition 5.3 is called the *hull-kernel-topology.*

We from now on equip $\mathrm{Prim}(\mathcal{S})$ with the hull-kernel-topology.

Proposition 5.5. *The following assertions are valid.*

(i) *The mappings*

$$\begin{aligned}\{\emptyset \neq A \subseteq \mathrm{Prim}(\mathcal{S})\,\text{closed}\} &\leftrightarrow \mathrm{Rad}(\mathcal{S})\\ A &\mapsto \ker(A)\\ \mathrm{hull}(I) &\mapsfrom I\end{aligned}$$

are mutually inverse bijections.

(ii) *The sets*

$$U_f := \{p \in \mathrm{Prim}(\mathcal{S}) \mid f \notin p\}$$

for $f \in \mathrm{C}(K)$ *define a base for the hull-kernel-topology of* $\mathrm{Prim}(\mathcal{S})$*.*

(iii) *If* K *is metrizable, then* $\mathrm{Prim}(\mathcal{S})$ *has a countable base.*

(iv) *The space* $\mathrm{Prim}(\mathcal{S})$ *is* T_0*. Given* $p \in \mathrm{Prim}(\mathcal{S})$*, the set* $\{p\}$ *is closed if and only if* p *is a maximal* $\mathcal{S}$*-ideal.*

(v) *The space* $\mathrm{Prim}(\mathcal{S})$ *is quasi-compact.*

(vi) *The mapping*

$$\pi\colon \mathrm{ex}\,\mathrm{P}_{\mathcal{S}}(K) \longrightarrow \mathrm{Prim}(\mathcal{S}), \quad \mu \mapsto I_\mu$$

is continuous and surjective.

Proof. Assertion (i) is obvious. For (ii) observe that $\mathrm{Prim}(\mathcal{S}) = U_{\mathbb{1}}$. Now take a closed set $\emptyset \neq A \subseteq \mathrm{Prim}(\mathcal{S})$. Then $A = \ker(I)$ for some $\mathcal{S}$-ideal I and we obtain

$$\mathrm{Prim}(\mathcal{S}) \setminus A = \bigcup_{f \in I} \{p \in \mathrm{Prim}(\mathcal{S}) \mid f \notin p\} = \bigcup_{f \in I} U_f.$$

Moreover, each U_f is open since $\mathrm{Prim}(\mathcal{S}) \setminus U_f = \mathrm{hull}(\{f\})$. This proves (ii). Assertion (iii) is a direct consequence of (ii).

We proceed with part (iv) and prove that $\mathrm{Prim}(\mathcal{S})$ is a T_0-space. If $p_1, p_2 \in \mathrm{Prim}(\mathcal{S})$ with $p_1 \neq p_2$, then $M_1 \neq M_2$ for the supports $M_i := \mathrm{supp}\, p_i$, $i = 1, 2$. We may assume that there is $x \in M_2 \setminus M_1$ and find $f \in \mathrm{C}(K)$ with $f|_{M_1} = 0$ and $f(x) = 1$. Then $p_1 \notin U_f$ and $p_2 \in U_f$.
For the second part of (iv) take a maximal $\mathcal{S}$-ideal and assume that $p \in \overline{\{m\}}$. Then $m \subseteq p$ and thus $m = p$ by maximality of m.
Conversely, suppose that $\{m\}$ is closed and take a maximal $\mathcal{S}$-ideal p with $m \subseteq p$. Then $\ker(\{m\}) = m \subseteq p$ and thus

$$p \in \mathrm{hull}(\ker(\{m\})) = \overline{\{m\}},$$

i.e., $p = m$.

For the proof of (v) take closed subsets $A_j \subseteq \mathrm{Prim}(\mathcal{S})$ for $j \in J$ with

$$\bigcap_{j \in J} A_j = \emptyset$$

and let $I_j := \ker(A_j)$ be the corresponding radical ideals for $j \in J$. We show that

$$\sum_{j \in J} I_j = \mathrm{C}(K).$$

Denote the ideal on the left side by I and assume that it is a proper invariant ideal. Since there are no dense ideals in $\mathrm{C}(K)$, the closure $\overline{I}$ is contained in a maximal $\mathcal{S}$-ideal p. But then $p \in A_j$ for each $j \in J$ since the sets A_j are closed, a contradiction.

Take $j_1, ..., j_k$ with $1 \in I_{j_1} + ... + I_{j_k}$ for some $k \in \mathbb{N}$. Then

$$\sum_{m=1}^{k} I_{j_m} = \mathrm{C}(K)$$

and consequently

$$\bigcap_{m=1}^{k} A_{j_m} = \emptyset.$$

Finally, assertion (vi) follows from the fact that the set

$$\pi^{-1}(U_f) = \{\mu \in \operatorname{ex} \mathrm{P}_{\mathcal{S}}(K) \mid \langle |f|, \mu \rangle \neq 0\}$$

is open in $\operatorname{ex} \mathrm{P}_{\mathcal{S}}(K)$ for each $f \in \mathrm{C}(K)$. □

Corollary 5.6. *A net $(p_\alpha)_{\alpha \in A}$ in $\operatorname{Prim}(\mathcal{S})$ converges to $p \in \operatorname{Prim}(\mathcal{S})$ if and only if for each open set $U \subseteq K$ with $\operatorname{supp} p \cap U \neq \emptyset$ there is $\alpha_0 \in A$ with $\operatorname{supp} p_\alpha \cap U \neq \emptyset$ for every $\alpha \geqslant \alpha_0$.*

Proof. Consider the sets $V_f := \{x \in K \mid f(x) \neq 0\}$ for $f \in \mathrm{C}(K)$. Proposition 5.5 (ii) shows that a net $(p_\alpha)_{\alpha \in A}$ in $\operatorname{Prim}(\mathcal{S})$ converges to $p \in \operatorname{Prim}(\mathcal{S})$ if and only if for each $f \in \mathrm{C}(K)$ with $f|_{\operatorname{supp} p} \neq 0$ there is $\alpha_0 \in A$ with $f|_{\operatorname{supp}(p_\alpha)} \neq 0$ for every $\alpha \geqslant \alpha_0$, i.e., for each $f \in \mathrm{C}(K)$ with $V_f \cap \operatorname{supp} p \neq \emptyset$ there is α_0 with $V_f \cap \operatorname{supp} p_\alpha \neq \emptyset$ for every $\alpha \geqslant \alpha_0$. Since the sets V_f are a base of the topology of K, this shows the claim. □

Example 5.7. (i) For the trivial semigroup $\mathcal{S} = \{\mathrm{Id}\}$ every ideal is invariant and $\operatorname{Prim}(\mathcal{S})$ coincides with the maximal ideal space of the commutative C*-algebra $\mathrm{C}(K)$, i.e., it is homeomorphic to K.

(ii) Consider the torus $K = \mathbb{T} = \{z \in \mathbb{C} \mid |z| = 1\}$ and the rotation $\varphi_a(z) := az$ for $z \in \mathbb{T}$ and some fixed $a \in \mathbb{T}$ with $a^k = 1$ for some $k \in \mathbb{N}$. Denote the group of kth roots of unity by G_k. The ergodic measures are then precisely the measures $\mu_b \in \mathrm{P}_{\mathcal{S}_\varphi}(K)$ with $\mu_b := \frac{1}{k} \sum_{j=0}^{k-1} \delta_{a^j b}$ for $b \in \mathbb{T}$. Their supports are clearly the sets

$$M_b := bG_k = \{bz \mid z \in \mathbb{T},\, z^k = 1\}$$

for $b \in \mathbb{T}$. Using Corollary 5.6 a moment's thought reveals that

$$\mathbb{T}/G_k \longrightarrow \operatorname{Prim}(\mathcal{S}_\varphi), \quad bG_k \mapsto I_{M_b}$$

is a homeomorphism if we endow the factor group $\mathbb{T}/G_k$ with the quotient topology.

(iii) Consider the space $L := \{0,1\}^{\mathbb{N}}$ and the shift $\varphi\colon L \longrightarrow L$ given by $\varphi((x_n)_{n\in\mathbb{N}}) := (x_{n+1})_{n\in\mathbb{N}}$ for each $(x_n)_{n\in\mathbb{N}} \in L$. For each $k \in \mathbb{N}$ consider the minimal non-empty closed invariant set

$$M_k := \{\varphi^n(x^k) \mid n \in \{0, ..., 2k-1\}\}$$

with $x^k = (x_m^k)_{m\in\mathbb{N}}$ defined by

$$x_m^k := \begin{cases} 0 & \text{if } m \in \{1, ..., k\} + 2k\mathbb{N}_0, \\ 1 & \text{else.} \end{cases}$$

Now if

$$K := \overline{\bigcup_{k\in\mathbb{N}} M_k},$$

then it is readily seen that K is the invariant set

$$\bigcup_{k\in\mathbb{N}} M_k \cup \{(x_m)_{m\in\mathbb{N}} \in L \mid (x_m)_{m\in\mathbb{N}} \text{ increasing or decreasing}\}.$$

We restrict φ to K and claim that $\mathrm{Prim}(\mathcal{S}_\varphi)$ is not Hausdorff. It suffices to show that the sequence $(m_n)_{n\in\mathbb{N}}$ in $\mathrm{Prim}(\mathcal{S}_\varphi)$ with $m_n := I_{M_n}$ converges to two different points.
To this end, consider $k \in \mathbb{N}$ and the open subset

$$U := \left(\prod_{i=1}^{k}\{1\} \times \prod_{i=k+1}^{\infty}\{0,1\}\right) \cap K.$$

Then $M_l \cap U \neq \emptyset$ for each $l \geqslant k$. By Remark 5.6 this implies $m_l \to I_{\{(1)_{n\in\mathbb{N}}\}}$ and a similar argument shows $m_l \to I_{\{(0)_{n\in\mathbb{N}}\}}$.

6. Continuous Functions on the Primitive Spectrum

It is our goal to describe the continuous functions on $\mathrm{Prim}(\mathcal{S})$. As above we write $\mathrm{M}(\mathcal{S})$ for the support of $\mathrm{rad}_{\mathcal{S}}(0)$, i.e., the closure of the union of all supports of invariant ergodic measures, and recall that the semigroup on $\mathrm{C}(\mathrm{M}(\mathcal{S}))$ induced by $\mathcal{S}$ is denoted by $\mathcal{S}_{\mathrm{rad}_{\mathcal{S}}(0)}$. Now consider the following functions.

Definition 6.1. For a function $f \in \mathrm{fix}(\mathcal{S}_{\mathrm{rad}_{\mathcal{S}}(0)})$ we define

$$\hat{f} \colon \mathrm{Prim}(\mathcal{S}) \longrightarrow \mathbb{C}, \quad I_\mu \mapsto \int_{\mathrm{M}(\mathcal{S})} f \,\mathrm{d}\mu.$$

Note that each $f \in \mathrm{fix}(\mathcal{S}_{\mathrm{rad}_{\mathcal{S}}(0)})$ is constant on supports of ergodic measures and therefore $\int_{\mathrm{M}(\mathcal{S})} f \,\mathrm{d}\mu$ only depends on I_μ and not on μ itself. Thus $\hat{f}$ is in fact well-defined for each $f \in \mathrm{fix}(\mathcal{S}_{\mathrm{rad}_{\mathcal{S}}(0)})$ and the next lemma shows that it is even continuous.

Lemma 6.2. *If $f \in \mathrm{fix}(\mathcal{S}_{\mathrm{rad}_{\mathcal{S}}(0)})$, then $\hat{f} \in \mathrm{C}(\mathrm{Prim}(\mathcal{S}))$.*

Proof. Let $p = I_\mu \in \mathrm{Prim}(\mathcal{S})$ and $\varepsilon > 0$. We set

$$f_\varepsilon := \sup\left(\varepsilon \cdot \mathbb{1} - \left| f - \int_{\mathrm{M}(\mathcal{S})} f \,\mathrm{d}\mu \cdot \mathbb{1}\right|, 0\right)\Big|_{\mathrm{M}(\mathcal{S})} \in \mathrm{C}(\mathrm{M}(\mathcal{S})).$$

Then $U := U_{f_\varepsilon}$ is an open neighborhood of p. Moreover, for each $q = I_\nu \in U$

$$\varepsilon - \left| \int_{\mathrm{M}(\mathcal{S})} f \,\mathrm{d}\nu - \int_{\mathrm{M}(\mathcal{S})} f \,\mathrm{d}\mu \right| > 0$$

which means $|\hat{f}(p) - \hat{f}(q)| < \varepsilon$. □

In the radical free case (i.e., $\mathrm{M}(\mathcal{S}) = K$) we now obtain a linear mapping from the fixed space $\mathrm{fix}(\mathcal{S})$ to $\mathrm{C}(\mathrm{Prim}(\mathcal{S}))$. It turns out that this is actually an isomorphism.

Theorem 6.3. *If $\mathcal{S}$ is radical free, then the fixed space* $\operatorname{fix}(\mathcal{S})$ *is a Banach sublattice of* $\mathrm{C}(K)$ *and the mapping*

$$\hat{}\colon \operatorname{fix}(\mathcal{S}) \longrightarrow \mathrm{C}(\operatorname{Prim}(\mathcal{S})), \quad f \mapsto \hat{f}$$

is an isometric Markov lattice isomorphism.

Proof. We first show that $\operatorname{fix}(\mathcal{S})$ is a sublattice of $\mathrm{C}(K)$. Take $f \in \operatorname{fix}(\mathcal{S})$ and $S \in \mathcal{S}$. Since $\mathrm{M}(\mathcal{S}) = K$, it suffices to prove that $S_p|f||_{\operatorname{supp} p} = |f||_{\operatorname{supp} p}$ for each $p \in \operatorname{Prim}(\mathcal{S})$. However, this is true since the fixed space $\operatorname{fix}(\mathcal{S}_p)$ consists only of constant functions for every $p \in \operatorname{Prim}(\mathcal{S})$.

The mapping $\hat{}$ is clearly linear and $\hat{\mathbb{1}} = \mathbb{1}$. Next we show that $\hat{}$ is an isometric Markov lattice homomorphism. For $f \in \operatorname{fix}(\mathcal{S})$

$$\|\hat{f}\| = \sup_{p \in \operatorname{Prim}(\mathcal{S})} |\hat{f}(p)| = \sup_{\mu \in \operatorname{ex} \mathrm{P}_{\mathcal{S}}(K)} |\langle f, \mu \rangle| \leqslant \|f\|.$$

The set $\{x \in K \mid |f(x)| = \|f\|\}$ is the support of an $\mathcal{S}$-ideal (see Theorem 1.2 in [Sin68]) and thus contains a minimal support of an $\mathcal{S}$-ideal M which in turn supports an ergodic measure μ. This implies

$$|\hat{f}(I_M)| = |\langle f, \mu \rangle| = |f(x)| = \|f\|$$

for each $x \in M$ and consequently $\|\hat{f}\| = \|f\|$.
Now take $f \in \operatorname{fix}(\mathcal{S})$ and $p = I_\mu \in \operatorname{Prim}(\mathcal{S})$. For each $x \in \operatorname{supp}(\mu)$

$$|\hat{f}(p)| = |f(x)| = |f|(x) = \widehat{|f|}(p).$$

It remains to show that $\hat{}$ is surjective. Take $f \in \mathrm{C}(\operatorname{Prim}(\mathcal{S}))$ with $0 \leqslant f \leqslant \mathbb{1}$. We fix $n \in \mathbb{N}$ and consider the open sets

$$U_{k,n} := \left\{ p \in \operatorname{Prim}(\mathcal{S}) \,\middle|\, \frac{k-1}{n} < f(p) < \frac{k+1}{n} \right\}$$

for $k \in \{0, ..., n\}$. Then $U_{k,n}^c = \operatorname{hull}(I_{k,n})$ for invariant ideals $I_{k,n} \subseteq \mathrm{C}(K)$ and $k \in \{0, ..., n\}$. Assume

$$I := \sum_{k=0}^{n} I_{k,n} \neq \mathrm{C}(K).$$

Then I is contained in a maximal $\mathcal{S}$-ideal p. Since $I_{k,n} \subseteq p$ for all $k \in \{0, ..., n\}$,

$$p \in \bigcap_{k=0}^{n} \operatorname{hull}(I_{k,n}) = \left(\bigcup_{k=0}^{n} U_{k,n} \right)^c = \emptyset,$$

a contradiction.
We thus find $0 \leqslant f_{k,n} \in I_{n,k}$ for $k \in \{0, ..., n\}$ with $\mathbb{1} = \sum_{k=0}^{n} f_{k,n}$ (see II.5.1.4 in [Bla06]). Now set $g_n := \sum_{k=1}^{n} \frac{k}{n} f_{k,n}$.
Take $p \in \operatorname{Prim}(\mathcal{S})$. If $k \in \{0, ..., n\}$ with $p \notin U_{k,n}$, then $I_{k,n} \subseteq p$ and therefore

$f_{k,n} \in p$, i.e., $f_{k,n}|_{\operatorname{supp} p} = 0$. This implies

$$\begin{aligned}|g_n(x) - f(p)| &= \left| \sum_{k\colon p \in U_{k,n}} \frac{k}{n} f_{k,n}(x) - \sum_{k\colon p\in U_{k,n}} f_{k,n}(x) f(p) \right| \\ &\leqslant \sum_{k\colon p\in U_{k,n}} \left| \frac{k}{n} - f(p) \right| |f_{k,n}(x)| \leqslant \frac{1}{n}\end{aligned}$$

for $x \in \operatorname{supp} p$. In particular we obtain

$$|g_n(x) - g_m(x)| \leqslant \frac{1}{n} + \frac{1}{m}$$

for all $x \in M := \bigcup_{p\in\operatorname{Prim}(\mathcal{S})} \operatorname{supp} p$ and all $n, m \in \mathbb{N}$. Since $\mathcal{S}$ is radical free, M is dense whence $(g_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathrm{C}(K)$. Denoting its limit by g we obtain $g(x) = f(p)$ for $x \in \operatorname{supp} p$, $p \in \operatorname{Prim}(\mathcal{S})$ and, since M is dense, $g \in \operatorname{fix}(\mathcal{S})$. Moreover, we clearly have $\hat{g} = f$. □

The first part of the proof of Theorem 6.3 is based on the proof of Theorem 5 in [Sch68], while the second part uses arguments from the Dauns-Hofmann Theorem II.6.5.10 in [Bla06].
We now focus on the general case, i.e., $\mathcal{S}$ not being radical free.

Lemma 6.4. *The mapping*

$$\vartheta\colon \operatorname{Prim}(\mathcal{S}) \longrightarrow \operatorname{Prim}(\mathcal{S}_{\operatorname{rad}_{\mathcal{S}}(0)}), \quad p \mapsto p|_{\mathrm{M}(\mathcal{S})}$$

is a homeomorphism.

Proof. Note first that ϑ is well-defined and bijective by Lemma 3.6 since $\operatorname{rad}_{\mathcal{S}}(0) \subseteq p$ for each $p \in \operatorname{Prim}(\mathcal{S})$. Lemma 3.6 also implies that

$$\operatorname{Rad}(\mathcal{S}) \longrightarrow \operatorname{Rad}(\mathcal{S}_{\operatorname{rad}_{\mathcal{S}}(0)}), \quad I \mapsto I|_{\mathrm{M}(\mathcal{S})}$$

is bijective. By Proposition 5.5 (i) we obtain that $A \subseteq \operatorname{Prim}(\mathcal{S})$ is closed if and only if $A = \operatorname{hull}(I)$ for some $I \in \operatorname{Rad}(\mathcal{S})$. If A is closed, we therefore obtain

$$\begin{aligned}\vartheta(A) &= \vartheta(\{p \in \operatorname{Prim}(\mathcal{S}) \mid I \subseteq p\}) \\ &= \{q \in \operatorname{Prim}(\mathcal{S}_{\operatorname{rad}_{\mathcal{S}}(0)}) \mid I \subseteq \vartheta^{-1}(q)\} \\ &= \{q \in \operatorname{Prim}(\mathcal{S}_{\operatorname{rad}_{\mathcal{S}}(0)}) \mid I|_{\mathrm{M}(\mathcal{S})} \subseteq q\} \\ &= \operatorname{hull}(I|_{\mathrm{M}(\mathcal{S})}),\end{aligned}$$

and therefore $\vartheta(A)$ is closed. Conversely, if $\vartheta(A)$ is closed, then by Lemma 3.6 and Proposition 5.5 (i) there is $I \in \operatorname{Rad}(\mathcal{S})$ with $\vartheta(A) = \operatorname{hull}(I|_{\mathrm{M}(\mathcal{S})})$ and, by the above, we obtain $A = \operatorname{hull}(I)$. □

By combining Lemma 6.4 with Theorem 6.3 we obtain the main result of this section.

Theorem 6.5. *The fixed space* $\operatorname{fix}(\mathcal{S}_{\operatorname{rad}_{\mathcal{S}}(0)})$ *is a Banach sublattice of* $\mathrm{C}(\mathrm{M}(\mathcal{S}))$ *and the mapping*

$$\hat{}\colon \operatorname{fix}(\mathcal{S}_{\operatorname{rad}_{\mathcal{S}}(0)}) \longrightarrow \mathrm{C}(\operatorname{Prim}(\mathcal{S})), \quad f \mapsto \hat{f}$$

is an isometric Markov lattice isomorphism.

7. Mean Ergodic Semigroups of Markov Operators

Using the space $\mathrm{C}(\mathrm{Prim}(\mathcal{S}))$ we can now analyze mean ergodicity of Markov semigroups and extend Theorem 2 of [Sch67]. Recall that $\overline{\mathrm{co}}\,\mathcal{S}$ denotes the closed convex hull of $\mathcal{S}$ with respect to the strong operator topology. The right amenable semigroup $\mathcal{S}$ is *mean ergodic* if there is $P \in \overline{\mathrm{co}}\,\mathcal{S}$ with $PS = SP = P$ for each $S \in \mathcal{S}$ (see [Nag73] or [Sch13] for this concept). In this case P is unique with these properties and a projection onto the fixed space $\mathrm{fix}(\mathcal{S})$ of $\mathcal{S}$, called the *mean ergodic projection*.

Theorem 7.1. *The following assertions are equivalent.*

(a) $\mathcal{S}$ *is mean ergodic.*

(b) *The following two conditions are satisfied.*

  (i) *The mapping*

$$\mathrm{ex}\,\mathrm{P}_{\mathcal{S}}(K) \longrightarrow \mathrm{Prim}(\mathcal{S}), \quad \mu \mapsto I_\mu$$

  is a homeomorphism.

  (ii) *For each* $f \in \mathrm{fix}(\mathcal{S}_{\mathrm{rad}_{\mathcal{S}}(0)})$ *there is* $F \in \mathrm{fix}(\mathcal{S})$ *with* $f = F|_{\mathrm{M}(\mathcal{S})}$.

(c) *The following three conditions are satisfied.*

  (i) *The primitive spectrum* $\mathrm{Prim}(\mathcal{S})$ *is a Hausdorff space.*

  (ii) *For each* $\mu \in \mathrm{ex}\,\mathrm{P}_{\mathcal{S}}(K)$ *the support* $\mathrm{supp}\,\mu$ *is uniquely ergodic, i.e.,* μ *is the only invariant measure having its support in* $\mathrm{supp}\,\mu$.

  (iii) *For each* $f \in \mathrm{fix}(\mathcal{S}_{\mathrm{rad}_{\mathcal{S}}(0)})$ *there is* $F \in \mathrm{fix}(\mathcal{S})$ *with* $f = F|_{\mathrm{M}(\mathcal{S})}$.

Remark 7.2. Note that assertion (c) (i) of Theorem 7.1 implies that each primitive ideal is maximal (see Proposition 5.5 (iv)). This shows that the concept of maximal $\mathcal{S}$-ideals is sufficient for mean ergodic semigroups. In particular, combining Theorem 7.1 with Proposition 3.7 (i) implies Theorem 2 of [Sch67].

Proof (of Theorem 7.1)**.** "(a) $\Rightarrow$ (c)": Assume that $\mathcal{S}$ is mean ergodic with mean ergodic projection $P \in \mathscr{L}(\mathrm{C}(K))$. We first show that $\mathrm{Prim}(\mathcal{S})$ is Hausdorff.
Consider $I_{\mu_1}, I_{\mu_2} \in \mathrm{Prim}(\mathcal{S})$ with $\mu_1 \neq \mu_2$. Since $\mathcal{S}$ is mean ergodic, we find $f \in \mathrm{fix}(\mathcal{S})$ with

$$c_1 := \langle f, \mu_1 \rangle < \langle f, \mu_2 \rangle =: c_2$$

by Theorem 1.7 of [Sch13]. Choose $c \in (c_1, c_2)$ and set $U_1 := f^{-1}((-\infty, c))$ and $U_2 := f^{-1}((c, \infty))$. The sets

$$V_i := \{p \in \mathrm{Prim}(\mathcal{S}) \mid \mathrm{supp}\, p \cap U_i \neq \emptyset\} \subseteq \mathrm{Prim}(\mathcal{S})$$

are open by Proposition 5.5 (ii) and $I_{\mu_i} \in V_i$ for $i = 1, 2$.
Assume there is $p \in V_1 \cap V_2$. Then there are $x_i \in U_i \cap \mathrm{supp}\, p$ for $i = 1, 2$ and thus $f(x_1) < c < f(x_2)$. Since f is constant on supports of ergodic measures, this is a contradiction.
Given $\mu \in \mathrm{ex}\,\mathrm{P}_{\mathcal{S}}(K)$ we know that $\mathcal{S}_{I_\mu} \subseteq \mathscr{L}(\mathrm{C}(\mathrm{supp}\,\mu))$ is also mean ergodic. Since $\mathrm{fix}(\mathcal{S}_{I_\mu})$ is one-dimensional, we obtain that $\mathrm{fix}(\mathcal{S}'_{I_\mu})$ is one dimensional, too. Thus, the supports of ergodic measures are uniquely ergodic.

Next, take $f \in \operatorname{fix}(\mathcal{S}_{\mathrm{rad}_{\mathcal{S}}(0)})$ and let G be any continuous extension of f to K. Then $F := PG \in \operatorname{fix}(\mathcal{S})$ with $F|_{\mathrm{M}(\mathcal{S})} = PG|_{\mathrm{M}(\mathcal{S})} = f$. Thus (a) implies (c).

"(c) $\Rightarrow$ (b)": Now suppose that (i), (ii) and (iii) of (c) are valid. We first show that $\operatorname{ex}\mathrm{P}_{\mathcal{S}}(K)$ is compact. Take a net $(\mu_\alpha)_{\alpha\in A}$ in $\operatorname{ex}\mathrm{P}_{\mathcal{S}}(K)$ with $\lim_\alpha \mu_\alpha = \mu \in \mathrm{P}_{\mathcal{S}}(K)$. Since I_μ is a radical ideal by Proposition 3.7 (i) we obtain

$$I_\mu = \bigcap_{\substack{p\in\mathrm{Prim}(\mathcal{S})\\ I_\mu\subseteq p}} p.$$

Now take $p \in \mathrm{Prim}(\mathcal{S})$ with $p \supseteq I_\mu$ and any $f \in \mathrm{C}(K)$ with $p \in U_f$. Then $f \notin p$ and consequently $f \notin I_\mu$. This implies $\langle |f|, \mu\rangle \neq 0$ and thus there is $\alpha_0 \in A$ with $\langle |f|, \mu_\alpha\rangle \neq 0$ for all $\alpha \geqslant \alpha_0$. But then $I_{\mu_\alpha} \to p$ and, since $\mathrm{Prim}(\mathcal{S})$ is Hausdorff, this implies that there is only one such p, hence I_μ is primitive. Applying (ii) shows that μ is ergodic.

We now obtain that

$$\pi\colon \operatorname{ex}\mathrm{P}_{\mathcal{S}}(K) \longrightarrow \mathrm{Prim}(\mathcal{S}), \quad \mu \mapsto I_\mu$$

is a homeomorphism since the mapping is injective by (ii), $\operatorname{ex}\mathrm{P}_{\mathcal{S}}(K)$ is compact and $\mathrm{Prim}(\mathcal{S})$ is Hausdorff by (i).

"(b) $\Rightarrow$ (a)": Finally assume that (b) is valid. The mapping

$$\Phi_1\colon \mathrm{C}(\mathrm{Prim}(\mathcal{S})) \longrightarrow \mathrm{C}(\operatorname{ex}\mathrm{P}_{\mathcal{S}}(K)), \quad f \mapsto f\circ\pi$$

is then an isometric Markov lattice isomorphism and by Theorem 6.5 the mapping

$$\hat{\ }\colon \operatorname{fix}(\mathcal{S}_{\mathrm{rad}_{\mathcal{S}}(0)}) \longrightarrow \mathrm{C}(\mathrm{Prim}(\mathcal{S}))$$

is so, too. Now consider the map

$$\Phi_2\colon \operatorname{fix}(\mathcal{S}) \longrightarrow \operatorname{fix}(\mathcal{S}_{\mathrm{rad}_{\mathcal{S}}(0)}), \quad f \mapsto f|_{\mathrm{M}(\mathcal{S})}.$$

This is an isometric Banach space embedding (isometry follows with the same arguments as in the proof of Theorem 6.5) and by (ii) it is surjective. Thus

$$\Phi := \Phi_1 \circ \hat{\ } \circ \Phi_2\colon \operatorname{fix}(\mathcal{S}) \longrightarrow \mathrm{C}(\operatorname{ex}\mathrm{P}_{\mathcal{S}}(K)), \quad f \mapsto \langle f, \cdot\rangle$$

is an isometric isomorphism of Banach spaces.

Now take $\mu_1, \mu_2 \in \mathrm{P}_{\mathcal{S}}(K)$ with $\mu_1 \neq \mu_2$. The space $\operatorname{ex}\mathrm{P}_{\mathcal{S}}(K)$ is compact since it is homeomorphic to $\mathrm{Prim}(\mathcal{S})$. By Choquet theory (see Proposition 1.2 in [Phe01]) we thus find measures $\tilde{\mu}_1, \tilde{\mu}_2 \in \mathrm{C}(\operatorname{ex}\mathrm{P}_{\mathcal{S}}(K))'$ with $\tilde{\mu}_1 \neq \tilde{\mu}_2$ such that

$$\langle f, \mu_i\rangle = \int_{\operatorname{ex}\mathrm{P}_{\mathcal{S}}(K)} \langle f, \nu\rangle \,\mathrm{d}\tilde{\mu}_i(\nu)$$

for each $f \in \mathrm{C}(K)$ and $i = 1, 2$. We then obtain

$$\langle f, \mu_i\rangle = \int_{\operatorname{ex}\mathrm{P}_{\mathcal{S}}(K)} \Phi(f)(\nu) \,\mathrm{d}\tilde{\mu}_i(\nu) = \langle \Phi(f), \tilde{\mu}_i\rangle$$

for each $f \in \operatorname{fix}(\mathcal{S})$ and $i = 1, 2$. Since $\mathrm{C}(\operatorname{ex}\mathrm{P}_{\mathcal{S}}(K))$ separates $\mathrm{C}(\operatorname{ex}\mathrm{P}_{\mathcal{S}}(K))'$, this proves that $\operatorname{fix}(\mathcal{S})$ separates $\mathrm{P}_{\mathcal{S}}(K)$. Now $\mathcal{S}$ consists of Markov operators and therefore $\operatorname{fix}(\mathcal{S})$ separates $\operatorname{fix}(\mathcal{S}')$. Thus $\mathcal{S}$ is mean ergodic by Theorem 1.7 of [Sch13].

□

Corollary 7.3. *If $\mathcal{S}$ is radical free, then $\mathcal{S}$ is mean ergodic if and only if*

$$\operatorname{ex} \mathrm{P}_{\mathcal{S}}(K) \longrightarrow \operatorname{Prim}(\mathcal{S}), \quad \mu \mapsto I_\mu$$

is a homeomorphism.

The next corollary follows from Proposition 2.9 and Lemma 6.4.

Corollary 7.4. *The semigroup $\mathcal{S}$ is mean ergodic if and only if $\mathcal{S}_{\operatorname{rad}_{\mathcal{S}}(0)}$ is mean ergodic and for each $f \in \operatorname{fix}(\mathcal{S}_{\operatorname{rad}_{\mathcal{S}}(0)})$ there is $F \in \operatorname{fix}(\mathcal{S})$ with $f = F|_{\mathrm{M}(\mathcal{S})}$.*

Finally we discuss some examples showing that the conditions of Theorem 7.1 (c) are independent of each other.

Example 7.5. Consider the following continuous mappings $\varphi\colon K \longrightarrow K$ and the induced semigroups $\mathcal{S} = \mathcal{S}_\varphi \subseteq \mathscr{L}(\mathrm{C}(K))$.

(i) If $K = [0,1]$ and $\varphi(x) = x^2$ for $x \in K$, then $\mathrm{M}(\mathcal{S}) = \{0,1\}$ and the primitive spectrum is the two point discrete space. Clearly, both fixed points define uniquely ergodic sets, so conditions (i) and (ii) of Theorem 7.1 (c) are fulfilled. However, $\mathcal{S}$ is not mean ergodic since the function $f\colon \{0,1\} \longrightarrow \mathbb{C}$ defined by $f(0) := 0$ and $f(1) := 1$ has no invariant continuous extension to K.

(ii) If $K = \mathbb{T}$ then there is a homeomorphism $\varphi\colon \mathbb{T} \longrightarrow \mathbb{T}$ such that $\mathcal{S}$ is not uniquely ergodic, but minimal (see Theorem 5.8 of [Par81]). In particular, the support of every ergodic measure is K. Thus the primitive spectrum is trivial and $\mathcal{S}$ is radical free whence Theorem 7.1 (c) (i) and (iii) are valid. $\mathcal{S}$ is not mean ergodic since supports of ergodic measures are not uniquely ergodic.

(iii) If K and φ are defined as in Example 5.7 (iii), then supports of ergodic measures are uniquely ergodic and $\mathcal{S}$ is radical free. Thus Theorem 7.1 (c) (ii) and (iii) are fulfilled. The semigroup $\mathcal{S}$ is not mean ergodic since the primitive spectrum is not Hausdorff.

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2 Submitted Manuscripts

2.1 Gelfand-type theorems for dynamical Banach modules

GELFAND-TYPE THEOREMS FOR DYNAMICAL BANACH MODULES

HENRIK KREIDLER AND SITA SIEWERT

Abstract. The representation theorems of Gelfand and Kakutani for commutative C*-algebras and AM- and AL-spaces are the basis for the Koopman linearization of topological and measure-preserving dynamical systems. In this article we prove versions of these results for dynamics on topological and measurable Banach bundles and the corresponding weighted Koopman representations on Banach modules.

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1. Introduction

The concept of *Koopman linearization* provides a very powerful method to study dynamical systems, see [EFHN15]. Given a *topological G-dynamical system*, i.e., a locally compact group G acting continuously on a locally compact space Ω, one can consider the induced *Koopman representation* of G as automorphisms of the commutative C*-algebra $\mathrm{C}_0(\Omega)$ of all continuous functions on Ω vanishing at infinity given by $T(g)f(x) := f(g^{-1}x)$ for $x \in \Omega$, $f \in \mathrm{C}_0(\Omega)$, and $g \in G$.

Passing to these linear operators opens the door for the use of functional analytic tools (e.g., spectral theory) to investigate the qualitative properties of the G-dynamical system. This is justified by Gelfand's representation theory which shows that no relevant information is lost in this process.

More precisely and in terms of category theory (see [Lan98] for an introduction), assigning the Koopman representation to a group action defines an equivalence of the category of topological G-dynamical systems and the category of strongly continuous representations of G as automorphisms of commutative C*-algebras, see, e.g., Section 1.4 of [Dix77] and Sections 4.3 and 4.4 of [EFHN15].

Likewise, in the measure theoretic setting Koopman representations on L^1-spaces reflect the qualitative behavior of measure-preserving systems up to null sets (under some separability assumptions, see Section 7.3 and Chapter 12 of [EFHN15]). Using Kakutani's representation theorem for AL-spaces (see Theorem II.8.5 of [Sch74]), such Koopman representations can also be characterized in terms of Banach lattice theory.

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These results connect topological dynamics and ergodic theory with functional analysis and operator theory and led, amongst others, to the classical and recent ergodic theorems.
In this article we prove suitable versions of these representation theorems for dynamics on Banach bundles and modules. This can be the starting point for a systematic operator theoretic investigation of differentiable flows on manifolds and their differentials on tangent bundles.

We consider a Banach bundle E over a locally compact or measure space X and dynamics on E compatible with a fixed group action on X. These *dynamical Banach bundles* then induce *weighted Koopman representations* on Banach spaces of sections of the bundle.
Such dynamical Banach bundles and the induced weighted Koopman representations appear naturally in many contexts. Important examples are so-called evolution families solving nonautonomous Cauchy problems (see Section VI.9 of [EN00]) and derivatives of smooth flows on manifolds (see Chapter 5 of [BP13]).
The goal of this article is to characterize such weighted Koopman representations via abstract algebraic and lattice theoretic properties.
The correspondence between topological Banach bundles and certain kinds of Banach modules has established in the 70s and 80s of the last century (see, e.g., [HK77] and [DG83]). We extend these results to a dynamical setting and then also treat the measure theoretic case.

We start in Section 2 by recalling the concepts of topological and measurable Banach bundles and introduce dynamics on these bundles. Concrete examples motivate the abstract concepts.
In the third section we consider Banach modules as the natural operator theoretic counterparts of Banach bundles. We introduce dynamics on these modules and give a first characterization of these operators via a locality condition (see Theorem 3.10). In particular, dynamical topological and measurable Banach bundles induce such "dynamical Banach modules" (see Example 3.13 and Example 3.16).
As in the case of Banach lattices (see Sections II.7, II.8 and II.9 of [Sch74]) there are two important classes of Banach modules which are dual to each other: AM-modules and AL-modules.
In Subsection 4.1 we focus on AM-modules, which are known in the literature as (locally) convex Banach modules, see [HK77] or [Gie98], and prove our first main result: A Gelfand-type representation theorem for dynamical AM-modules (see Theorem 4.6). In Subsection 4.2 we then discuss the duality between AM- and AL-modules (see Proposition 4.17).
In Section 5 we see that AM- and AL-modules admit a lattice theoretic structure (see Proposition 5.3 and Proposition 5.9). In Theorem 5.5 and Theorem 5.16 we show that the algebraic structure of a module and this lattice theoretic structure are strongly related. In particular, weighted Koopman operators can be characterized algebraically (as *weighted module homomorphisms*) or in a lattice theoretic way (as *dominated operators*).
We use the lattice theoretic structure to prove our second representation

theorem, which clarifies the relation between dynamical measurable Banach bundles and AL-modules (see Theorem 5.12). It should be pointed out that—in contrast to the "AM case"—even the non-dynamical version of this result seems to be new (see Proposition 5.18).

In the following all vector spaces are over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ and all locally compact spaces are Hausdorff.

2. Dynamical Banach bundles

2.1. **The topological case.** In this section we define dynamics on topological Banach bundles over some fixed topological dynamical system. Recall the following abstract definition of a Banach bundle (see Definition 1.1 in [DG83], see also [HK77]).

Definition 2.1. A *(topological) Banach bundle* over a locally compact space Ω is a pair (E, p_E) consisting of a topological space E and a continuous, open and surjective mapping $p_E \colon E \longrightarrow \Omega$ satisfying the following conditions.

(i) Each fiber $E_x := p_E^{-1}(x)$ for $x \in \Omega$ is a Banach space.

(ii) The mappings

$$+\colon E \times_\Omega E \longrightarrow E, \quad (u,v) \mapsto u +_{E_{p_E(v)}} v,$$
$$\cdot\colon \mathbb{K} \times E \longrightarrow E, \quad (\lambda, v) \mapsto \lambda \cdot_{E_{p_E(v)}} v$$

are continuous where $E \times_\Omega E := \bigcup_{x\in\Omega} E_x \times E_x \subseteq E \times E$ is equipped with the subspace topology.

(iii) The map

$$\|\cdot\|\colon E \longrightarrow \mathbb{R}_{\geqslant 0}, \quad v \mapsto \|v\|_{E_{p_E(v)}}$$

is upper semicontinuous.

(iv) For each $x \in \Omega$ and each open set $W \subseteq E$ containing the zero $0_x \in E_x$ there exist $\varepsilon > 0$ and an open neighborhood U of x such that

$$\{v \in p_E^{-1}(U) \mid \|v\| \leqslant \varepsilon\} \subseteq W.$$

In the following we usually suppress the mapping p_E and denote the bundle (E, p_E) simply by E. Moreover, we call E a *continuous Banach bundle* if the mapping $\|\cdot\|$ is continuous.

Remark 2.2. Note that if E is a Banach bundle over a locally compact space Ω, we obtain a Banach bundle $\tilde{E}$ over the one-point compactfication $K := \Omega \cup \{\infty\}$ in a canonical way by taking the space $\tilde{E} := E \cup \{0\}$, the canonical mapping $p_{\tilde{E}} \colon \tilde{E} \longrightarrow K$ and the topology on $\tilde{E}$ generated by the topology on E and the sets

$$U(L, \varepsilon) := \{v \in p_{\tilde{E}}^{-1}(\Omega \setminus L) \mid \|v\| < \varepsilon\}$$

for compact $L \subseteq \Omega$ and $\varepsilon > 0$. In the following we will frequently make use of this fact.

We now list some important examples of Banach bundles.

Example 2.3. (i) Let Z be any Banach space and Ω a locally compact space. Then $E := \Omega \times Z$ is a continuous Banach bundle over Ω, called the *trivial bundle with fiber* Z if $p_E\colon \Omega \times Z \longrightarrow K$ is the projection onto the first component and $\Omega \times Z$ is equipped with the product topology.

(ii) Consider a Riemannian manifold M. Then the tangent bundle TM over M is a continuous Banach bundle over M.

(iii) Let $\pi\colon L \longrightarrow K$ be a continuous surjection between compact spaces L and K. For each $k \in K$ let $L_k := \pi^{-1}(k)$ be the associated fiber. We define

$$E := \dot{\bigcup_{k \in K}} \mathrm{C}(L_k),$$
$$p_E\colon E \longrightarrow K, \quad v \in \mathrm{C}(L_k) \mapsto k$$

and endow this with the topology generated by the sets

$$W(s, U, \varepsilon) := \{v \in p_E^{-1}(U) \mid \|v - s|_{L_{p(h)}}\|_{\mathrm{C}(L_{p(h)})} < \varepsilon\}$$

where $U \subseteq K$ is open, $s \in \mathrm{C}(L)$ and $\varepsilon > 0$. Then (E, p_E) is a Banach bundle over K. Moreover, it is easy to see that E is a continuous Banach bundle if and only if π is open. This construction has been used in topological dynamics (see e.g., page 30 of [Kna67] or Section 5 of [Ell87]).

The topology of a Banach bundle is determined by its continuous sections. We make this precise by the following definition and the subsequent lemma.

Definition 2.4. Let E be a Banach bundle over a locally compact space Ω. A continuous mapping $s\colon \Omega \longrightarrow E$ is a *continuous section* of E if $p_E \circ s = \mathrm{id}_\Omega$. We write $\Gamma(E)$ for the space of continuous sections of E and

$$\Gamma_0(E) := \{s \in \Gamma(E) \mid \forall\, \varepsilon > 0\, \exists \text{ compact } K \subseteq \Omega \text{ with } \|s(x)\| \leqslant \varepsilon\, \forall\, x \notin K\}$$

for the subspace of all *continuous sections vanishing at infinity.*

Lemma 2.5. *Let E be a Banach bundle over a locally compact space Ω. For $v \in E$ the sets*

$$V(s, U, \varepsilon) := \{w \in E \mid p_E(w) \in U, \|w - s(p_E(w))\| < \varepsilon\},$$

with $s \in \Gamma_0(E)$ satisfying $s(p_E(v)) = v$, $U \subseteq \Omega$ an open neighborhood of $p(v)$ and $\varepsilon > 0$, form a neighborhood base of v in E.

Proof. In the case of a compact base space this follows from 1.5 and 3.16 of [Gie98]. The general case can readily be reduced to this by considering $\tilde{E}$ (cf. Remark 2.2). □

In order to define dynamics on Banach bundles we need morphisms between them (cf. page 17 of [DG83]).

Definition 2.6. Let Ω be a locally compact space and $\varphi\colon \Omega \longrightarrow \Omega$ a continuous mapping. Consider Banach bundles E and F over Ω. A *(bounded)*

Banach bundle morphism over φ from E to F is a continuous mapping

$$\Phi\colon E \longrightarrow F$$

such that

(i) $p_F \circ \Phi = \varphi \circ p_E$, i.e., the diagram

$$\begin{array}{ccc} E & \xrightarrow{\Phi} & F \\ {\scriptstyle p_E}\downarrow & & \downarrow{\scriptstyle p_F} \\ \Omega & \xrightarrow{\varphi} & \Omega \end{array}$$

commutes,

(ii) $\Phi|_{E_x} \in \mathscr{L}(E_x, F_{\varphi(x)})$ for each $x \in \Omega$,

(iii) $\|\Phi\| := \sup_{x\in\Omega} \|\Phi|_{E_x}\|_{\mathscr{L}(E_x,F_{\varphi(x)})} < \infty$.

Moreover, Φ is *isometric* if $\Phi|_{E_x}$ is an isometry for each $x \in \Omega$. If $\varphi = \mathrm{id}_\Omega$, we simply call a Banach bundle morphism over φ a *Banach bundle morphism*.

We are interested in dynamical Banach bundles over invertible dynamical systems. Therefore we fix a *topological G-dynamical system* $(\Omega; \varphi)$ for the rest of the section, i.e., Ω is assumed to be a locally compact space and G is a locally compact group acting on Ω via the continuous mapping

$$\varphi\colon G \times \Omega \longrightarrow \Omega, \quad (g, x) \mapsto \varphi_g(x) = gx.$$

Moreover, let $S \subseteq G$ be a closed subsemigroup of G containing the neutral element e, i.e., a submonoid of G. Important examples of this situation are the cases of $G = \mathbb{Z}$, $S = \mathbb{N}_0$ and $G = \mathbb{R}$, $S = \mathbb{R}_{\geqslant 0}$.

Definition 2.7. An *S-dynamical Banach bundle over* $(\Omega; \varphi)$ is a pair $(E; \Phi)$ of a Banach bundle E over Ω and a monoid representation

$$\Phi\colon S \longrightarrow E^E, \quad g \mapsto \Phi_g,$$

such that

(i) the mapping

$$\Phi_g\colon E \longrightarrow E$$

is a Banach bundle morphism over φ_g for each $g \in S$,

(ii) Φ is *jointly continuous*, i.e., the mapping

$$S \times E \longrightarrow E, \quad (g, v) \mapsto \Phi_g(v)$$

is continuous,

(iii) Φ is *locally bounded*, i.e., $\sup_{g\in K} \|\Phi_g\| < \infty$ for every compact subset $K \subseteq S$.

A *morphism* from an S-dynamical Banach bundle $(E; \Phi)$ over $(\Omega; \varphi)$ to an S-dynamical Banach bundle $(F; \Psi)$ over $(\Omega; \varphi)$ is a Banach bundle morphism

$\Theta\colon E \longrightarrow F$ such that the diagram

$$\begin{tikzcd} E \arrow[r, "\Theta"] \arrow[d, "\Phi_g"'] & F \arrow[d, "\Psi_g"] \\ E \arrow[r, "\Theta"'] & F \end{tikzcd}$$

commutes for each $g \in S$.

Remark 2.8. If $\Omega = K$ is compact, then conditions (i) and (ii) of Definition 2.7 already imply (iii). This can be seen using the same arguments as in the proof of Proposition 1.4 of [DG83].

Remark 2.9. The concept of a dynamical Banach bundle is closely related to the notion of cocycles and linear skew-product flows (cf. Definition 6.1 of [CL99]). In fact, if $(E;\Phi)$ is an S-dynamical Banach bundle over $(\Omega;\varphi)$, the operators $\Phi_{g,x} := \Phi_g|_{E_x} \in \mathscr{L}(E_x, E_{\varphi_g(x)})$ for $g \in S$ and $x \in K$ satisfy the *cocycle rule*

$$\Phi_{g_1 g_2,x} = \Phi_{g_1,\varphi_{g_2}(x)} \circ \Phi_{g_2,x}$$

for all $g_1, g_2 \in S$ and $x \in \Omega$.

Now we consider dynamics on the Banach bundles of Example 2.3.

Example 2.10. (i) Assume that $G = \mathbb{R}$, $S = \mathbb{R}_{\geqslant 0}$, Z is a Banach space and $E = \Omega \times Z$ is the corresponding trivial Banach bundle.
If $\{\Phi^t(x) \in \mathscr{L}(Z) \mid x \in \Omega, t \geqslant 0\}$ is a strongly continuous exponentially bounded cocycle in the sense of Definition 6.1 of [CL99], then the continuous linear skew-product flow Φ_t given by

$$\Phi_t(x,v) := (\varphi_t(x), \Phi^t(x)v)$$

for $x \in \Omega$, $v \in Z$ and $t \geqslant 0$ defines an $\mathbb{R}_{\geqslant 0}$-dynamical Banach bundle $(E;\Phi)$ over $(\Omega;\varphi)$. Conversely, each $\mathbb{R}_{\geqslant 0}$-dynamical Banach bundle $(E;\Phi)$ defines a strongly continuous exponentially bounded cocycle by setting

$$\Phi^t(x)v := \mathrm{pr}_2(\Phi_t(x,v))$$

for $x \in \Omega$, $v \in Z$ and $t \geqslant 0$, where $\mathrm{pr}_2\colon \Omega \times Z \longrightarrow Z$ is the projection onto the second component.
In particular, evolution families (see Example 6.5 of [CL99] and Section IV.9 of [EN00]) define $\mathbb{R}_{\geqslant 0}$-dynamical Banach bundles.

(ii) If $\Omega = M$ is a Riemannian manifold and $\varphi_g\colon M \longrightarrow M$ is differentiable for each $g \in G$, then, by the chain rule, the differentials $\mathrm{D}\varphi_g$ define a G-dynamical Banach bundle over $(M;\varphi)$.

(iii) Assume that $\Omega = K$ is compact and $\pi\colon (L;\psi) \longrightarrow (K;\varphi)$ is an extension of topological G-dynamical systems, i.e., a continuous surjection intertwining the dynamics, and E is defined as in Example 2.3 (iii). For each $g \in G$ consider

$$\Phi_g\colon E \longrightarrow E, \quad v \in \mathrm{C}(L_k) \mapsto v \circ \psi_{g^{-1}} \in \mathrm{C}(L_{\varphi_g(k)}).$$

This defines a G-dynamical Banach bundle $(E;\Phi)$ over $(K;\varphi)$.

2.2. **The measurable case.** A measure space X is a triple $(\Omega_X, \Sigma_X, \mu_X)$ consisting of a set Ω_X, a σ-algebra Σ_X of subsets of Ω_X and a positive σ-finite measure $\mu_X\colon \Sigma_X \longrightarrow [0,\infty]$. We also assume that our measure spaces are *complete*, i.e., subsets of null sets are measurable.
We define Banach bundles over measure spaces as in Section II.4 of [FD88] or Appendix A.3 of [ADR00] (see also [Gut93b]).

Definition 2.11. A *(measurable) Banach bundle* over a measure space X is a triple $(E, p_E, \mathcal{M}_E)$ where E is a set, $p_E\colon E \longrightarrow \Omega_X$ is a surjective mapping such that the fiber $E_x := p_E^{-1}(x)$ is a Banach space for each $x \in \Omega_X$ and $\mathcal{M}_E$ is a linear subspace of

$$\mathcal{S}_E := \{s\colon \Omega_X \longrightarrow E \mid p_E \circ s = \mathrm{id}_{\Omega_X}\}$$

such that

(i) if $f\colon \Omega_X \longrightarrow \mathbb{K}$ is measurable and $s \in \mathcal{M}_E$, then $fs \in \mathcal{M}_E$, where
$$fs\colon s \longrightarrow E, \quad x \mapsto f(x)s(x),$$

(ii) for each $s \in \mathcal{M}_E$ the mapping
$$|s|\colon \Omega_X \longrightarrow \mathbb{R}_{\geqslant 0}, \quad x \mapsto \|s(x)\|_{E_x}$$
is measurable,

(iii) if $(s_n)_{n\in\mathbb{N}}$ is a sequence in $\mathcal{M}_E$ converging almost everywhere to $s \in \mathcal{S}_E$, then $s \in \mathcal{M}_E$.

Elements $s \in \mathcal{S}_E$ are called *sections* and elements $s \in \mathcal{M}_E$ are called *measurable sections.*
The bundle is *separable* if, in addition,

(iv) there is a sequence $(s_n)_{n\in\mathbb{N}}$ in $\mathcal{M}_E$ such that $\mathrm{lin}\{s_n(x) \mid n \in \mathbb{N}\}$ is dense in E_x for almost every $x \in \Omega_X$.

We mostly just write E for a measurable Banach bundle $(E, p_E, \mathcal{M}_E)$.

Remark 2.12. Let X be a measure space and (E, p_E) a pair such that E is a set and $p_E\colon E \longrightarrow \Omega_X$ is a surjective mapping such that the fiber $E_x := p_E^{-1}(x)$ is a Banach space for each $x \in \Omega_X$. Then by Section II.4.2 of [FD88] every linear subspace $\mathcal{M}_E$ of $\mathcal{S}_E$ satisfying condition (iii) of Definition 2.11 *generates* a measurable Banach bundle, i.e., there is a smallest linear subspace $\tilde{\mathcal{M}}_E$ of $\mathcal{S}_E$ such that $(E, p_E, \tilde{\mathcal{M}}_E)$ is a measurable Banach bundle. Moreover, $\tilde{\mathcal{M}}_E$ consists precisely of all almost everywhere limits of sequences in $\mathrm{lin}\{\mathbb{1}_A s \mid A \in \Sigma_X, s \in \mathcal{M}_E\}$.

We briefly list some examples for measurable Banach bundles and refer to Appendix A.3 of [ADR00] for additional examples.

Example 2.13. (i) Let X be a measure space and Z a Banach space. Consider $E := X \times Z$ with the projection p_E onto the first component. The space of sections $\mathcal{S}_E$ can be identified with the space of all functions from X to Z. The set of all strongly measurable functions (see Section 1.3.5 of [HP57]) then defines a subset $\mathcal{M}_E$ of $\mathcal{S}_E$ which turns E into a measurable Banach bundle called the *trivial Banach bundle with fiber Z*. This coincides with the measurable

Banach bundle generated by the constant sections (see Section II.5.1 of [FD88]).

(ii) Let E be a topological Banach bundle over a locally compact space Ω, μ be a regular Borel probability measure on Ω and $\mathcal{B}(\Omega)$ the Borel σ-algebra of Ω. Then the space $\Gamma_0(E)$ (see Definition 2.4) generates a measurable Banach bundle E_μ over the completion of the measure space $(\Omega, \mathcal{B}(\Omega), \mu)$. See Section 15 of [FD88] for a more explicit description of the measurable sections of a continuous Banach bundle.

Before introducing dynamics on measurable Banach bundles, we first define morphisms of measure spaces. A *premorphism* $\varphi\colon X \longrightarrow Y$ between measure spaces X and Y is a measurable and measure-preserving mapping $\varphi\colon \Omega_X \longrightarrow \Omega_Y$. Setting $\varphi \sim \psi$ if $\varphi(x) = \psi(x)$ for almost every $x \in \Omega_X$ defines an equivalence relation on the set of premorphisms from X to Y. The equivalence classes with respect to this equivalence relation are then the *morphisms* from X to Y. As usual, given a morphism we will implicitly choose a representative of it but also denote it by φ when there is no room for confusion.
We now define morphisms of measurable Banach bundles in a similar manner.

Definition 2.14. Let $\varphi\colon X \longrightarrow X$ be a morphism on a measure space X. Consider Banach bundles E and F over X. A *premorphism* Φ from E to F over φ is a mapping $\Phi\colon E \longrightarrow F$ such that

(i) $\Phi \circ \mathcal{M}_E \subseteq \mathcal{M}_F \circ \varphi$,

(ii) $p_F \circ \Phi = \varphi \circ p_E$ almost everywhere,

(iii) $\Phi|_{E_x} \in \mathscr{L}(E_x, F_{\varphi(x)})$ for almost every $x \in \Omega_X$,

(iv) $\|\Phi\| := \operatorname{ess\,sup}_{x\in\Omega_X} \|\Phi|_{E_x}\| < \infty$.

Again, we want to identify premorphisms which agree up to a null set. Set

$$\mathrm{Premor}_\varphi(E,F) := \{\Phi\colon E \longrightarrow F \text{ premorphism over } \varphi\},$$
$$\mathcal{N}_\varphi(E,F) := \{\Phi \in \mathrm{Premor}_\varphi(E,F) \mid \Phi = 0 \text{ almost everywhere}\},$$

and $\mathrm{Mor}_\varphi(E,F) := \mathrm{Premor}_\varphi(E,F)/\mathcal{N}_\varphi(E,F)$ for measurable Banach bundles E and F as above.
An equivalence class $[\Phi] \in \mathrm{Mor}_\varphi(E,F)$ is called a *morphism of measurable Banach bundles over* φ. It is *isometric* if $\Phi|_{E_x}$ is isometric for almost every $x \in \Omega_X$. If $\varphi = \mathrm{id}_X$, we call a morphism over φ simply a *morphism of measurable Banach bundles.*
As above, we will implicitly choose representatives of morphisms whenever necessary and denote them with the same symbol.

Now we introduce dynamical measurable Banach bundles. For the rest of this section let G be a group with neutral element $e \in G$. A *measure-preserving G-dynamical system* $(X;\varphi)$ is a measure space X together with a group homomorphism

$$\varphi\colon G \longrightarrow \mathrm{Aut}(X), \quad g \mapsto \varphi_g,$$

where $\mathrm{Aut}(X)$ is the set of automorphisms of X. Also fix a submonoid $S \subseteq G$.

Definition 2.15. An *S-dynamical Banach bundle over* $(X;\varphi)$ is a pair $(E;\Phi)$ of a measurable Banach bundle E over X and a monoid representation

$$\Phi\colon S \longrightarrow E^E, \quad g \mapsto \Phi_g$$

such that Φ_g is a morphism over φ_g for every $g \in S$. We call $(E;\Phi)$ *separable* if E is separable.

A *morphism* between measurable Banach bundles $(E;\Phi)$ and $(F;\Psi)$ over $(X;\varphi)$ is a morphism $\Theta\colon E \longrightarrow F$ of Banach bundles such that the diagram

$$\begin{tikzcd} E \arrow[r, "\Theta"] \arrow[d, "\Phi_g"'] & F \arrow[d, "\Psi_g"] \\ E \arrow[r, "\Theta"'] & F \end{tikzcd}$$

commutes for each $g \in S$.

Example 2.16. (i) Let E be the trivial bundle with fiber Z (see Example 2.13 (i)). Then the S-dynamical Banach bundles correspond to *measurable cocycles*, i.e., mappings $\Phi\colon S \times X \longrightarrow \mathscr{L}(Z)$ such that

- $\Phi(gh, x) = \Phi(g, \varphi_h(x)) \circ \Phi(h, x)$ for almost every $x \in X$ for all $g, h \in S$,
- $\Phi(e, x) = \mathrm{Id}_Z$ for almost every $x \in X$,
- $X \longrightarrow Z$, $x \mapsto \Phi(g, x)v$ is strongly measurable for all $g \in S$ and $v \in Z$,
- $\operatorname{ess\,sup}_{x\in\Omega_X} \|\Phi(g, x)\| < \infty$ for every $g \in S$.

(ii) Let $(E;\Phi)$ be a topological S-dynamical Banach bundle over a topological G-dynamical system $(\Omega;\varphi)$ (with G and S discrete) and let μ be a regular Borel measure on Ω. Moreover, let E_μ be the induced measurable Banach bundle of Example 2.13 (ii). Then $(E_\mu;\Phi)$ is an S-dynamical measurable Banach bundle.

3. Dynamical Banach modules

In the previous sections we have defined dynamics on topological and measurable Banach bundles. We now consider Banach modules as the operator theoretic counterparts. First we recall the following definition from Section 2 of [DG83].

Definition 3.1. Let A be a commutative C*-algebra. A Banach space Γ which is also an A-module is a *Banach module over* A if $\|fs\| \leqslant \|f\|\|s\|$ for all $f \in A$ and $s \in \Gamma$.

A *homomorphism* from a Banach module Γ over A to a Banach module Λ over A is a bounded operator $T \in \mathscr{L}(\Gamma, \Lambda)$ which is also an A-module homomorphism. It is *isometric* if T is an isometry.

In the following we always assume that Banach modules Γ over a commutative C*-algebra A are *non-degenerate* (see [Par08]) in the sense that

$$\Gamma = \overline{\mathrm{lin}}\,\{fs \mid f \in A, s \in \Gamma\}.$$

Note that if $(e_i)_{i\in I}$ is an approximate unit for A (see Section 1.8 of [Dix77]), then this is the case if and only if $\lim_i e_i s = s$ for each $s \in \Gamma$. In particular, if A has a unit, then the module is unitary.

We now discuss Banach modules associated with Banach bundles.

Example 3.2. Let E be a topological Banach bundle over a locally compact space Ω. Then $\Gamma_0(E)$ (see Definition 2.4) is a Banach module over $\mathrm{C}_0(\Omega)$ if equipped with the operation

$$\mathrm{C}_0(\Omega) \times \Gamma_0(E) \longrightarrow \Gamma_0(E), \quad (f, s) \mapsto [x \mapsto f(x)s(x)]$$

and the norm $\|\cdot\|$ defined by $\|s\| := \sup_{x\in\Omega} \|s(x)\|$ for $s \in \Gamma_0(E)$.

Remark 3.3. Let Ω be a locally compact space and E a Banach bundle over Ω. If K is the one-point compactification of Ω and $\tilde{E}$ the extended bundle of E (see Remark 2.2), then

$$\Gamma(\tilde{E}) \to \Gamma_0(E), \quad s \mapsto s|_\Omega$$

is an isometric isomorphism of Banach spaces. In particular, we can consider $\Gamma_0(E)$ as a Banach module over $\mathrm{C}(K)$.

Example 3.4. For a measurable Banach bundle E over a measure space X we define

$$\begin{aligned}
\mathcal{N}_E &:= \{s \in \mathcal{M}_E \mid s = 0 \text{ almost everywhere}\},\\
\Gamma^1(E) &:= \left\{s \in \mathcal{M}_E \mid |s| \text{ is integrable}\right\}/\mathcal{N}_E,\\
\Gamma^\infty(E) &:= \left\{s \in \mathcal{M}_E \mid |s| \text{ is essentially bounded}\right\}/\mathcal{N}_E.
\end{aligned}$$

With the natural norms and operations the spaces $\Gamma^1(E)$ and $\Gamma^\infty(E)$ are Banach modules over $\mathrm{L}^\infty(X)$.

In order to define dynamical Banach modules we now proceed as above and define first "morphisms over morphisms".

Definition 3.5. Let A be a commutative C*-algebra and $T \in \mathscr{L}(A)$ a *-homomorphism. Moreover, let Γ and Λ be Banach modules over A. Then $\mathcal{T} \in \mathscr{L}(\Gamma, \Lambda)$ is a *T-homomorphism* if

$$\mathcal{T}(fs) = Tf \cdot \mathcal{T}s \text{ for all } f \in A \text{ and } s \in \Gamma.$$

Example 3.6. (i) Let $\varphi\colon \Omega \longrightarrow \Omega$ be a homeomorphism of a locally compact space Ω. Then the *Koopman operator* $T_\varphi \in \mathscr{L}(\mathrm{C}_0(\Omega))$ defined by $T_\varphi f := f \circ \varphi^{-1}$ for $f \in \mathrm{C}_0(\Omega)$ is a *-automorphism.
If E and F are Banach bundles over Ω and $\Phi\colon E \longrightarrow F$ is a Banach bundle morphism over φ, the *weighted Koopman operator* $\mathcal{T}_\Phi \in \mathscr{L}(\Gamma_0(E), \Gamma_0(F))$ given by $\mathcal{T}_\Phi s := \Phi \circ s \circ \varphi^{-1}$ for $s \in \Gamma_0(E)$ is a T_φ-homomorphism.

(ii) Let $\varphi\colon X \longrightarrow X$ be an automorphism of a measure space X. Then the *Koopman operator* $T_\varphi \in \mathscr{L}(\mathrm{L}^\infty(X))$ defined by $T_\varphi f := f \circ \varphi^{-1}$ for $f \in \mathrm{L}^\infty(X)$ is a *-automorphism.
If E and F are Banach bundles over X and $\Phi\colon E \longrightarrow F$ is a Banach bundle morphism over φ, the *weighted Koopman operator* $\mathcal{T}_\Phi \in \mathscr{L}(\Gamma^1(E), \Gamma^1(F))$ given by $\mathcal{T}_\Phi s := \Phi \circ s \circ \varphi^{-1}$ for $s \in \Gamma^1(E)$ is a T_φ-homomorphism. Similarly, Φ induces an operator $\mathcal{T}_\Phi \in \mathscr{L}(\Gamma^\infty(E), \Gamma^\infty(F))$.

Before introducing the concept of dynamical Banach modules we prove a different characterization of T-homomorphisms as some sort of "locality preserving operators". We start with the following definition.

Definition 3.7. Let A be a commutative C*-algebra and Γ a Banach module over A. For $s \in \Gamma$ we call the closed ideal

$$I_s := \{f \in A \mid fs = 0\}$$

the *supporting ideal of s in A*.

If $A = \mathrm{C}_0(\Omega)$ for some locally compact space Ω, there is a correspondence between the concept of supporting ideals and the following notion of support (see Definition 9.3 of [AAK92]).

Definition 3.8. Let Ω be a locally compact space and Γ a Banach module over $\mathrm{C}_0(\Omega)$. For $s \in \Gamma$ we call

$$\mathrm{supp}(s) := \{x \in \Omega \mid \text{each } f \in \mathrm{C}_0(\Omega) \text{ with } f(x) \neq 0 \text{ satisfies } fs \neq 0\} \subseteq \Omega$$

the *support of s in Ω*.

Lemma 3.9. *Let Ω be a locally compact space and Γ a Banach module over $\mathrm{C}_0(\Omega)$. Then*

$$I_s = \{f \in \mathrm{C}_0(\Omega) \mid f|_{\mathrm{supp}(s)} = 0\}.$$

for every $s \in \Gamma$.

Proof. Let $s \in \Gamma$. Since I_s is a closed ideal in $\mathrm{C}_0(\Omega)$, we find a unique closed subset M such that $f|_M = 0$ if and only if $f \in I_s$. It is clear that $\mathrm{supp}(s) \subseteq M$. On the other hand, if $x \in \Omega \setminus \mathrm{supp}(s)$, we find $f \in \mathrm{C}_0(\Omega)$ with $f(x) \neq 0$ but $fs = 0$. Then $f|_M = 0$ which shows $x \notin M$. □

The following is a first characterization of T-homomorphisms extending Theorem 9.5 of [AAK92].

Theorem 3.10. *Let Ω be a locally compact space, Γ and Λ Banach modules over $\mathrm{C}_0(\Omega)$ and $T \in \mathscr{L}(\mathrm{C}_0(\Omega))$ a *-automorphism. For $\mathcal{T} \in \mathscr{L}(\Gamma, \Lambda)$ the following assertions are equivalent.*

(a) *$\mathcal{T}$ is a T-homomorphism.*

(b) *$TI_s \subseteq I_{\mathcal{T}s}$ for every $s \in \Gamma$.*

(c) $\mathrm{supp}(\mathcal{T}s) \subseteq \varphi(\mathrm{supp}(s))$ *for each $s \in \Gamma$.*

For the proof we need the following lemma.

Lemma 3.11. *Let* Ω *be a locally compact space and* Γ *be a Banach module over* $\mathrm{C}_0(\Omega)$. *Let* $K = \Omega \cup \{\infty\}$ *be the one-point compactification of* Ω. *The mapping*

$$\mathrm{C}(K) \times \Gamma \longrightarrow \Gamma, \quad (f, s) \mapsto (f - f(\infty)\mathbb{1})|_\Omega s + f(\infty)s$$

turns Γ *into a (unitary) Banach module over* $\mathrm{C}(K)$.

Proof. It is easy to check that the mapping above actually turns Γ into a module over $\mathrm{C}(K)$. Choose an approximate unit $(e_i)_{i\in I}$ for $\mathrm{C}_0(\Omega)$. Now take $f \in \mathrm{C}(K)$ and $s \in \Gamma$ and observe that

$$\begin{aligned} \|fs\| &= \lim_i \|(f - f(\infty)\mathbb{1})|_\Omega e_i s + f(\infty)e_i s\| \\ &= \lim_i \|(fe_i)s\| \leqslant \limsup_i \|e_i f\|\|s\| \\ &\leqslant \|f\|\|s\|. \end{aligned}$$

This shows $\|fs\| \leqslant \|f\|\|s\|$ and therefore Γ is a Banach module over $\mathrm{C}(K)$. □

Proof (of Theorem 3.10)**.** The equivalence of (b) and (c) is obvious while the equivalence of (a) and (c) follows from Theorem 9.5 of [AAK92] if $K = \Omega$ is compact and $\varphi = \mathrm{id}_K$[1].

Now take Ω non-compact but still assume $\varphi = \mathrm{id}_\Omega$. We consider the one-point compactification K of Ω and the module structure of Γ over $\mathrm{C}(K)$ (see Lemma 3.11). For $s \in \Gamma$ we denote the support of s with respect to this module structure by $\mathrm{supp}_K(s)$. It is easy to see that

$$\overline{\mathrm{supp}(s)}^K \subseteq \mathrm{supp}_K(s) \subseteq \mathrm{supp}(s) \cup \{\infty\}.$$

Let $(e_i)_{i\in I}$ be an approximate unit for $\mathrm{C}_0(\Omega)$. It is easy to see that $\infty \notin \mathrm{supp}_K(s)$ if and only if there is $g \in \mathrm{C}_0(\Omega)$ with $gs = s$. But this is the case if and only if there is $i_0 \in A$ with $(e_i g - e_i)s = 0$, i.e., $(e_i g - e_i)|_{\mathrm{supp}(s)} = 0$ for every $i \geqslant i_0$. Therefore, the result for non-compact Ω can be reduced to the compact case.

Finally let $\varphi\colon \Omega \longrightarrow \Omega$ be an arbitrary homeomorphism of a locally compact space Ω. Consider the module Λ_{T_φ} which is the space Λ equipped with the new operation $f \cdot_{T_\varphi} s := T_\varphi f \cdot s$ for $f \in \mathrm{C}_0(\Omega)$ and $s \in \Lambda$. Then $\mathcal{T} \in \mathscr{L}(\Gamma, \Lambda)$ is a T_φ-homomorphism if and only if $\mathcal{T} \in \mathscr{L}(\Gamma, \Lambda_{T_\varphi})$ is a homomorphism of Banach modules. By the above, this is the case if and only if

$$\{x \in \Omega \mid \text{each } f \in \mathrm{C}_0(\Omega) \text{ with } f(x) \neq 0 \text{ satisfies } T_\varphi f \cdot \mathcal{T}s \neq 0\} \subseteq \mathrm{supp}(s),$$

i.e., $\mathrm{supp}(\mathcal{T}s) \subseteq \varphi(\mathrm{supp}(s))$ for each $s \in \Gamma$. □

We now introduce dynamical Banach modules. Fix a pair $(A; T)$ of a commutative C*-algebra A and a strongly continuous group representation $T\colon G \longrightarrow \mathscr{L}(A)$ of a locally compact group G as *-automorphisms of A. Moreover, let $S \subseteq G$ be a fixed closed submonoid.

Definition 3.12. An *S-dynamical Banach module over* $(A; T)$ is a pair $(\Gamma; \mathcal{T})$ consisting of a Banach A-module Γ and a monoid representation $\mathcal{T}\colon S \longrightarrow \mathscr{L}(\Gamma)$ such that

[1] Note that even though the authors work in the complex setting, their proof also works in the real case.

(i) $\mathcal{T}(g) \in \mathscr{L}(\Gamma)$ is a $T(g)$-homomorphism for each $g \in S$,

(ii) $\mathcal{T}$ is *strongly continuous*, i.e.,

$$S \longrightarrow \Gamma, \quad g \mapsto \mathcal{T}(g)s$$

is continuous for every $s \in \Gamma$.

A *homomorphism* from an S-dynamical Banach module $(\Gamma; \mathcal{T})$ over $(A; T)$ to an S-dynamical Banach module $(\Lambda; \mathcal{S})$ over $(A; T)$ is a homomorphism $V \in \mathscr{L}(\Gamma, \Lambda)$ of Banach modules over A such that the diagram

$$\begin{tikzcd} \Gamma \arrow[r, "V"] \arrow[d, "\mathcal{T}(g)"'] & \Lambda \arrow[d, "\mathcal{S}(g)"] \\ \Gamma \arrow[r, "V"'] & \Lambda \end{tikzcd}$$

commutes for each $g \in S$.

Starting with the topological case, we now show that dynamical Banach bundles induce dynamical Banach modules.

Example 3.13. Consider an S-dynamical Banach bundle $(E; \Phi)$ over a topological G-dynamical system $(\Omega; \varphi)$. For each $g \in G$ the Koopman operator $T_\varphi(g) := T_{\varphi_g}$ is a *-auotmorphism of $\mathrm{C}_0(\Omega)$ (see Example 3.6 (i)) and $g \mapsto T_\varphi(g)$ defines a representation of G as operators on $\mathrm{C}_0(\Omega)$, called the *Koopman representation* which is strongly continuous (this is probably well-known, but also a special case of Proposition 3.14 below).
By setting $\mathcal{T}_\Phi(g) := \mathcal{T}_{\Phi_g}$ for each $g \in S$ we obtain a $T_\varphi(g)$-homomorphism $\mathcal{T}_\Phi(g) \in \mathscr{L}(\Gamma_0(E))$ for each $g \in S$ (see Example 3.6). We call the monoid representation $\mathcal{T}_\Phi$ the *weighted Koopman representation* of $(E; \Phi)$.

Proposition 3.14. *Let $(\Omega; \varphi)$ be a topological G-dynamical system, $A = \mathrm{C}_0(\Omega)$ and $T = T_\varphi$ the Koopman representation of $(\Omega; \varphi)$.*

(i) *If $(E; \Phi)$ is an S-dynamical Banach bundle over $(\Omega; \varphi)$, then the weighted Koopman representation $\mathcal{T}_\Phi$ defines an S-dynamical Banach module over $(\mathrm{C}_0(\Omega); T_\varphi)$.*

(ii) *For a morphism $\Theta\colon (E; \Phi) \longrightarrow (F; \Psi)$ of S-dynamical Banach bundles over $(\Omega; \varphi)$ the operator $V_\Theta \in \mathscr{L}(\Gamma_0(E), \Gamma_0(F))$ defined by*

$$V_\Theta s := \Theta \circ s \quad \text{for } s \in \Gamma_0(E)$$

is a homomorphism $V_\Theta \in \mathscr{L}(\Gamma_0(E), \Gamma_0(F))$ between the S-dynamical Banach modules $(\Gamma_0(E); \mathcal{T}_\Phi)$ and $(\Gamma_0(F); \mathcal{T}_\Psi)$.

For the proof we need the following lemma.

Lemma 3.15. *Let $(E; \Phi)$ be an S-dynamical Banach bundle over $(\Omega; \varphi)$. Let $K := \Omega \cup \{\infty\}$ be the one-point compactification of Ω. Then the following assertions hold.*

(i) *The mapping*

$$\tilde{\varphi}\colon G \times K \longrightarrow K, \quad (g, x) \mapsto \begin{cases} \infty & x = \infty, \\ \varphi(g, x) & x \neq \infty, \end{cases}$$

is continuous.

(ii) *Setting*

$$\tilde{\Phi}\colon S \times \tilde{E} \longrightarrow \tilde{E}, \quad (g,v) \mapsto \begin{cases} 0 & v \in E_\infty, \\ \Phi_g v & v \in E, \end{cases}$$

defines an S*-dynamical Banach bundle* $(\tilde{E};\tilde{\Phi})$ *over* $(K;\tilde{\varphi})$*.*

Proof. If $g \in G$ and L is a compact subset of Ω, we choose a compact neighborhood V of g and set $U := (V^{-1} \cdot L)^c$. Then U is cocompact with $hy \notin L$ for all $h \in V$ and $y \in U$. This shows (i).

Now let $L \subseteq \Omega$ be compact, $\varepsilon > 0$ and $g \in S$. Choose a compact neighborhood $V \subseteq S$ of g and an open subset U in Ω with compact complement such that $hx \notin L$ for all $h \in V$ and $x \in U$. Since Φ is locally bounded, we find a $\delta > 0$ with $\|\Phi_h\| < \frac{1}{\delta}$ for every $h \in V$.

For $v \in E$ with $\|v\| < \delta\varepsilon$ and $p_E(v) \in U$ and $h \in V$ we then have $p_{\tilde{E}}(\Phi_h v) \notin L$ and $\|\Phi_h v\| < \varepsilon$. This shows that $\tilde{\Phi}$ is jointly continuous. ◻

Proof (of Proposition 3.14)**.** We first prove continuity of the weighted Koopman representation in the case of a compact space $\Omega = K$. Fix $s \in \Gamma(E)$ and let $g \in S$ and $\varepsilon > 0$. For each $x \in K$ the set

$$V := V(\Phi_g \circ s \circ \varphi_{g^{-1}}, K, \varepsilon) := \{v \in E \mid \|v - \Phi_g s(g^{-1}(p_E(v)))\| < \varepsilon\}$$

is a neighborhood of $\Phi_g s(g^{-1}x)$. Since the mapping

$$S \times K \longrightarrow E, \quad (h,y) \mapsto \Phi_h s(y)$$

is continuous as a composition of the continuous mappings

$$\begin{aligned} S \times K &\longrightarrow S \times E, \quad (h,y) \mapsto (h, s(y)), \\ S \times E &\longrightarrow E, \quad (h,v) \mapsto \Phi_h v, \end{aligned}$$

we find a neighborhood $O \subseteq S$ of g and a neighborhood $U \subseteq K$ of $g^{-1}x$ such that $\Phi_h s(y) \in V$ for every $h \in O$ and $y \in U$, i.e.,

$$\|\Phi_h s(y) - \Phi_g s(g^{-1}hy))\| < \varepsilon.$$

By compactness of K we thus find a neighborhood $W \subseteq S$ of g with

$$\sup_{y \in K} \|\Phi_h s(y) - \Phi_g s(g^{-1}hy))\| < \varepsilon$$

for all $h \in W$. But then

$$\sup_{y \in K} \|\Phi_h s(h^{-1}y) - \Phi_g s(g^{-1}y))\| = \sup_{y \in K} \|\Phi_h s(y) - \Phi_g s(g^{-1}hy))\| < \varepsilon$$

for each $h \in W$.

The general case of (i) now follows from Lemma 3.15 and Remark 3.3 and part (ii) is obvious. ◻

Example 3.16. Let G carry the discrete topology, $(X;\varphi)$ be a measure-preserving G-dynamical system, $A = \mathrm{L}^\infty(X)$ and $T = T_\varphi$ the induced *Koopman representation* on $\mathrm{L}^\infty(X)$, i.e., $T_\varphi(g) := T_{\varphi_g}$ for every $g \in G$.

Then every S-dynamical Banach bundle $(E;\Phi)$ over $(X;\varphi)$ induces a *weighted Koopman representation* $\mathcal{T}_\Phi$ on $\Gamma^1(E)$ which defines a dynamical Banach bundle. Moreover, if $\Theta\colon (E;\Phi) \longrightarrow (F;\Psi)$ is a morphism of S-dynamical Banach bundles over $(X;\varphi)$, then $V_\Theta s := \Theta \circ s$ for $s \in \Gamma^1(E)$ defines a homomorphism from $(\Gamma^1(E);\mathcal{T}_\Phi)$ to $(\Gamma^1(F);\mathcal{T}_\Psi)$.

4. AM- and AL-modules

We have seen that topological and measurable Banach bundles induce dynamical Banach modules and that these assignments are functorial. We now describe the essential ranges of these functors.
For this we recall a connection between Banach modules and Banach lattices, observed by Kaijser in Proposition 2.1 of [Kai78] and Abramovich, Arenson and Kitover in Lemma 4.6 of [AAK92] in the compact case. We give a new proof for the locally compact case based on Lemma 1 of [Cun67] and also provide more details on the lattice structure.

Proposition 4.1. *If Ω is a locally compact space, Γ a Banach module over $\mathrm{C}_0(\Omega)$ and $s \in \Gamma$, then the submodule $\Gamma_s := \overline{\mathrm{C}_0(\Omega) \cdot s}$ is a Banach lattice with positive cone $\overline{\mathrm{C}_0(\Omega)_+ \cdot s}$. Moreover, we obtain the following for $f, g \in \mathrm{C}_0(\Omega, \mathbb{R})$,*

(i) $fs \leqslant gs$ *if and only if* $f|_{\mathrm{supp}(s)} \leqslant g|_{\mathrm{supp}(s)}$,

(ii) $(fs \vee gs) = (f \vee g)s$,

(iii) $(fs \wedge gs) = (f \wedge g)s$,

(iv) $|fs| = |f|s$.

Proof. Since Γ_s is also a Banach module over $\mathrm{C}_0(\Omega)/I_s \cong \mathrm{C}_0(\mathrm{supp}(s))$ and the canonical mapping $\mathrm{C}_0(\Omega) \longrightarrow \mathrm{C}_0(\mathrm{supp}(s))$ is a lattice homomorphism (see Proposition II.2.6 of [Sch74]) we may assume that $I_s = \{0\}$.
Now let $f, g \in \mathrm{C}_0(\Omega)$ with $|g| \leqslant |f|$. Take $N := g^{-1}(\{0\})$ and choose an approximate unit $(e_i)_{i \in I}$ for $I_N := \{h \in \mathrm{C}_0(\Omega) \mid h|_N = 0\}$ such that e_i has compact support for every $i \in I$. Also define $h \in \mathrm{C}_0(\Omega)$ by

$$h_i(x) := \begin{cases} e_i(x)\frac{g(x)}{f(x)}, & x \notin N, \\ 0, & x \in N. \end{cases}$$

Then $|h_i(x)| \leqslant 1$ for every $x \in \Omega$ and therefore

$$\|gs\| = \lim_i \|e_i gs\| = \lim_i \|h_i fs\| \leqslant \limsup_i \|h_i\|\|fs\| \leqslant \|fs\|.$$

Now consider the case $\mathbb{K} = \mathbb{R}$ and equip the normed space $\mathrm{C}_0(\Omega) \cdot s$ with the order generated by the cone $\mathrm{C}_0(\Omega)_+ \cdot s$. Using that $I_s = \{0\}$, it follows that $fs \leqslant gs$ for $f, g \in \mathrm{C}_0(\Omega)$ if and only if $f \leqslant g$. This implies that $A \cdot s$ is a normed vector lattice with properties (i) – (iv) and by Corollary 2 on page 84 of [Sch74] Γ_s is a Banach lattice with positive cone $\overline{A_+ \cdot s}$. Moreover, since every normed vector lattice is a sublattice of its completion, the properties (i) – (iv) still hold with respect to the lattice operations of Γ_s.

Now take $\mathbb{K} = \mathbb{C}$. If $t \in \overline{\mathrm{C}_0(\Omega, \mathbb{R})s} \cap \mathrm{i}\overline{\mathrm{C}_0(\Omega, \mathbb{R})s}$, we find sequences $(f_n)_{n \in \mathbb{N}}$ and $(g_n)_{n \in \mathbb{N}}$ in $\mathrm{C}_0(\Omega)$ such that

$$t = \lim_{n \to \infty} f_n s = \lim_{n \to \infty} \mathrm{i} g_n s.$$

In particular, $\lim_{n\to\infty}(f_n - \mathrm{i}g_n)s = 0$. The inequality $|f_n| \leqslant |f_n - \mathrm{i}g_n|$ implies

$$\|f_n s\| \leqslant \|(f_n - \mathrm{i}g_n)s\|$$

for every $n \in \mathbb{N}$ and therefore $t = \lim_{n\to\infty} f_n s = 0$. Consequently, $\overline{\mathrm{C}_0(\Omega,\mathbb{R})s} \cap \mathrm{i}\overline{\mathrm{C}_0(\Omega,\mathbb{R})s} = \{0\}$.
On the other hand, if $t = \lim_{n\to\infty} f_n s$ for a sequence $(f_n)_{n\in\mathbb{N}}$, a similar argument shows that $((\mathrm{Re} f_n)\cdot s)_{n\in\mathbb{N}}$ and $((\mathrm{Im} f_n)\cdot s)_{n\in\mathbb{N}}$ are Cauchy sequences in Γ. This yields $t \in \overline{\mathrm{C}_0(\Omega,\mathbb{R})s} + \mathrm{i}\overline{\mathrm{C}_0(\Omega,\mathbb{R})s}$.
Thus $\Gamma_s = \overline{\mathrm{C}_0(\Omega,\mathbb{R})s} \oplus \mathrm{i}\overline{\mathrm{C}_0(\Omega,\mathbb{R})s}$ is the complexification of a real Banach lattice, hence a complex Banach lattice. □

We use this observation to introduce different types of Banach modules.

4.1. **AM-modules.** The first is based on the concept of AM-spaces (see [Sch74], Section II.7).

Definition 4.2. Let Ω be a locally compact space. A Banach module Γ over $\mathrm{C}_0(\Omega)$ is an *AM-module over* $\mathrm{C}_0(\Omega)$ if Γ_s is an AM-space for each $s \in \Gamma$.

Remark 4.3. Clearly a Banach module over $\mathrm{C}_0(\Omega)$ is an AM-module over $\mathrm{C}_0(\Omega)$ if and only if

$$\max(\|f_1 s\|, \|f_2 s\|) = \|(f_1 \vee f_2)s\|$$

for all $f_1, f_2 \in \mathrm{C}_0(\Omega)_+$ and $s \in \Gamma$.

Example 4.4. If E is a topological Banach bundle over a locally compact space Ω, then $\Gamma_0(E)$ (see Definition 2.4) is an AM-module over $\mathrm{C}_0(\Omega)$.

Remark 4.5.
(i) AM-modules are known in the literature as *locally convex Banach modules* (see Definition 7.10 in [Gie98] or Definition 1.1 of [Par08], see also [HK77]) and are defined differently. By Proposition 7.14 of [Gie98] our definition is equivalent in the unital case, and using an approximate identity, even in the general setting. Our terminology leads to a duality between AM and AL-modules, see Proposition 4.17 below.
(ii) Given a compact space K, each AM-module over $\mathrm{C}(K)$ is isometrically isomorphic to a space of sections $\Gamma(E)$ of some Banach bundle E over K which is unique up to isometric isomorphy (see Theorems 2.5 and 2.6 of [DG83]). The same holds (and is probably well-known) in the locally compact case if $\Gamma(E)$ is replaced with $\Gamma_0(E)$. However, since we did not find a reference for this fact, we give a proof in Proposition 4.10 below.

We now state and prove our first representation result for dynamical Banach modules.

Theorem 4.6. *Let G be a locally compact group, $S \subseteq G$ be a closed submonoid and $(\Omega;\varphi)$ a topological G-dynamical system. Then the assignments*

$$(E;\Phi) \mapsto (\Gamma_0(E); \mathcal{T}_\Phi)$$
$$\Theta \mapsto V_\Theta$$

define an essentially surjective, fully faithful functor from the category of S-dynamical topological Banach bundles over $(\Omega;\varphi)$ to the category of S-dynamical AM-modules over $(\mathrm{C}_0(\Omega);T_\varphi)$.

The proof of Theorem 4.6 starts with the following simple observation.

Lemma 4.7. *Let Ω be a locally compact space, $\varphi\colon \Omega \longrightarrow \Omega$ a homeomorphism and (E,p_E) be a Banach bundle over Ω. Then (E_φ,p_φ) with $E_\varphi := E$ and $p_\varphi := \varphi^{-1}\circ p_E$ is a Banach bundle over Ω which has the following properties.*

(i) *The identical mapping $\mathrm{id}_E\colon E \longrightarrow E_\varphi$ is a Banach bundle morphism over φ^{-1}.*

(ii) *If F is a Banach bundle over Ω, then a mapping $\Phi\colon F \longrightarrow E$ is a Banach bundle morphism over φ if and only if $\Phi\colon F \longrightarrow E_\varphi$ is a Banach bundle morphism over id_Ω.*

Using these facts, most of the proof of Theorem 4.6 can be reduced to the non-dynamical case. We first consider single operators.

Lemma 4.8. *Let E and F be Banach bundles over a locally compact space Ω. Moreover, let $\varphi\colon \Omega \longrightarrow \Omega$ be a homeomorphism and $\mathcal{T} \in \mathscr{L}(\Gamma_0(E),\Gamma_0(F))$ a T_φ-module homomorphism. Then there is a unique Banach bundle morphism Φ over φ with $\mathcal{T} = \mathcal{T}_\Phi$. Moreover, $\|\Phi\| = \|\mathcal{T}\|$ and $\mathcal{T}$ is an isometry if and only if Φ is isometric.*

Proof. Assume that $\Omega = K$ is compact. Consider the bundle F_φ induced by φ, see Lemma 4.7. The operator $V \in \mathscr{L}(\Gamma(E),\Gamma(F_\varphi))$ defined by $Vs := s\circ\varphi$ is an isometric and surjective $T_{\varphi^{-1}}$-homorphism. Therefore, $V\mathcal{T} \in \mathscr{L}(\Gamma(E),\Gamma(F_\varphi))$ is a (non-dynamical) homomorphism of Banach modules. By Theorem 2.6 of [DG83] we thus find a unique bundle morphism $\Phi\colon E \longrightarrow F_\varphi$ over id_K with

$$V\mathcal{T}s = \Phi\circ s$$

for each $s \in \Gamma(E)$, i.e., $\Phi\colon E \longrightarrow F$ is the unique bundle morphism over φ with

$$\mathcal{T}s = V^{-1}(\Phi\circ s) = \Phi\circ s\circ\varphi^{-1}$$

for every $s \in \Gamma(E)$. Moreover, $\|\Phi\| = \|V\mathcal{T}\| = \|\mathcal{T}\|$ and Φ is isometric if and only if $V\mathcal{T}$ is an isometry, i.e., if and only if $\mathcal{T}$ is isometric (see Propositions 10.13 and 10.16 of [Gie98]).

Now suppose that Ω is non-compact, but locally compact. Let K be the one-point compactification and $\tilde{\varphi}\colon K \longrightarrow K$ the canonical continuous extension of φ. The canonical mapping

$$\Gamma(\tilde{E}) \longrightarrow \Gamma_0(E), \quad s \mapsto s|_\Omega$$

is an isometric isomorphism of Banach spaces (see Remark 3.3) and therefore $\mathcal{T}$ induces an operator $\tilde{\mathcal{T}} \in \mathscr{L}(\Gamma(\tilde{E}),\Gamma(\tilde{F}))$. It is easy to check that $\tilde{\mathcal{T}}$ is a $T_{\tilde{\varphi}}$-homomorphism and we can apply the first part to find a unique bundle morphism $\tilde{\Phi}\colon \tilde{E} \longrightarrow \tilde{E}$ over $\tilde{\varphi}$ with $\mathcal{T}(s|_\Omega) = (\tilde{\Phi}\circ s\circ\tilde{\varphi}^{-1})|_\Omega$ for every $s \in \Gamma(\tilde{E})$. Since each Banach bundle morphism of E has a unique extension to a Banach bundle morphism of $\tilde{E}$, the restriction $\tilde{\Phi}|_E$ is the unique bundle morphism Φ over φ with $\mathcal{T}s := \Phi \circ s \circ \varphi^{-1}$ for all $s \in \Gamma_0(E)$. The remaining claims are obvious. $\square$

Lemma 4.9. *Let G be a locally compact group with identity element $e \in G$, $S \subseteq G$ be a closed submonoid and $(\Omega; \varphi)$ a topological G-dynamical system. Moreover, let E be a Banach bundle over Ω and let $\mathcal{T}\colon S \longrightarrow \mathscr{L}(\Gamma_0(E))$ be a strongly continuous monoid representation such that $(\Gamma_0(E); \mathcal{T})$ is an S-dynamical Banach module over $(\mathrm{C}_0(\Omega); T_\varphi)$. Then there is a unique S-dynamical Banach bundle $(E; \Phi)$ over $(\Omega; \varphi)$ such that $\mathcal{T}_\Phi = \mathcal{T}$.*

Proof. We apply Lemma 4.8 to find a unique bundle morphism Φ_g over φ_g such that $\mathcal{T}(g) = \mathcal{T}_{\Phi_g}$ for each $g \in S$. Since $\mathcal{T}(e) = \mathrm{Id}_{\Gamma_0(E)}$, we obtain that $\Phi(e) = \mathrm{id}_E$. Moreover, for $g_1, g_2 \in S$ we obtain that $\tilde{\Phi} := \Phi_{g_1} \circ \Phi_{g_2}$ is a bundle morphism over $\varphi_{g_1 g_2}$ with

$$\mathcal{T}(g_1 g_2) = \mathcal{T}(g_1)\mathcal{T}(g_2) = \mathcal{T}_\Phi(g_1)\mathcal{T}_\Phi(g_2) = \mathcal{T}_{\tilde{\Phi}}.$$

By uniqueness of $\Phi_{g_1 g_2}$ we therefore obtain

$$\Phi_{g_1} \circ \Phi_{g_2} = \tilde{\Phi} = \Phi_{g_1 g_2}.$$

To conclude the proof we have to show that the mapping

$$\Phi\colon S \longrightarrow E^E, \quad g \mapsto \Phi_g$$

is jointly continuous and that Φ is locally bounded. The latter follows since $\|\Phi(g)\| = \|\mathcal{T}(g)\|$ for every $g \in S$ by Lemma 4.8 and $\mathcal{T}$ is locally bounded by strong continuity and the principle of uniform boundedness.
Now let $v \in E$ and $g \in S$. Take $s \in \Gamma_0(E)$ with $s(gp_E(v)) = \Phi_g v$, $\varepsilon > 0$ and an open neighborhood $U \subseteq K$ of $gp_E(v)$. Since Φ_g is continuous, we find $\tilde{s} \in \Gamma_0(E)$, $\delta > 0$ and a neighborhood $\tilde{V}$ of $p_E(v)$ such that $\tilde{s}(p_E(v)) = v$ and

$$\Phi_g(V(\tilde{s}, \tilde{V}, \delta)) \subseteq V(s, U, \varepsilon),$$

in particular $\|\Phi_g \tilde{s}(x) - s(gx)\| < \varepsilon$ for every $x \in \tilde{V}$. Since φ is continuous, we find a neigborhood $V \subseteq \tilde{V}$ of $p_E(v)$ and a neighborhood $\tilde{W}$ of g in S such that $hy \in g(\tilde{V}) \cap U$ for every $y \in V$ and $h \in \tilde{W}$.
Finally choose a compact neighborhood $W \subseteq \tilde{W}$ of g with

$$\sup_{x \in \Omega} \|\Phi_h \tilde{s}(x) - \Phi_g \tilde{s}(g^{-1}hx)\| = \|\mathcal{T}(h)\tilde{s} - \mathcal{T}(g)\tilde{s}\| < \varepsilon.$$

for every $h \in W$. Then $M := \sup_{h \in W} \|\mathcal{T}(h)\| < \infty$ and for $h \in W$ and $u \in V(\tilde{s}, V, \frac{\varepsilon}{M+1})$, we obtain

$$\begin{aligned} \|\Phi_h u - s(hp_E(u))\| &\leqslant \|\Phi_h\| \cdot \|u - \tilde{s}(p_E(u))\| \\ &\quad + \|\Phi_h \tilde{s}(p_E(u)) - \Phi_g \tilde{s}(g^{-1}hp_E(u))\| \\ &\quad + \|\Phi_g \tilde{s}(g^{-1}hp_E(u)) - s(hp_E(u))\| \\ &< 3\varepsilon, \end{aligned}$$

and $hp_E(u) \in U$. This shows

$$\Phi_h u \in V(s, U, \varepsilon)$$

for each $h \in W$ and $u \in V(\tilde{s}, V, \frac{\varepsilon}{M+1})$ and thus Φ is jointly continuous. $\square$

Finally, we look at AM-modules.

Proposition 4.10. *Let Ω be a locally compact space and Γ an AM-module over* $\mathrm{C}_0(\Omega)$. *Then there is a Banach bundle E over Ω such that $\Gamma_0(E)$ is isometrically isomorphic to Γ. Moreover, this bundle is unique up to isometric isomorphy.*

Proof. If Ω is compact, the claim holds by Theorem 2.6 of [DG83]. If Ω is non-compact, we consider Γ as a Banach module over $\mathrm{C}(K)$ where K is the one-point compactification of Ω (see Lemma 3.11). Using a similar argument as in Lemma 3.11 we see that Γ is then an AM-module over $\mathrm{C}(K)$ and we therefore find a Banach bundle F over K such that $\Gamma(F)$ is isometrically isomorphic to Γ as a Banach module over $\mathrm{C}(K)$. Moreover, by the proof of Theorem 2.6 of [DG83] we have $F_\infty \cong \Gamma/J_\infty$ with

$$J_\infty = \overline{\mathrm{lin}}\{fs \mid f \in \mathrm{C}(K) \text{ with } f(\infty) = 0 \text{ and } s \in \Gamma\}.$$

Since Γ is non-degenerate, we obtain $J_\infty = \Gamma$ and thus $F_\infty = \{0\}$. We can therefore define a Banach bundle E over Ω by setting $E := F \setminus F_\infty$ and $p_E := p_F|_E$ and it is clear that $F = \tilde{E}$. In particular, we obtain an isometric isomorphism of Banach spaces (see Remark 3.3)

$$\Gamma(F) \longrightarrow \Gamma_0(E), \quad s \mapsto s|_\Omega$$

and it is then easy to check that Γ is isometrically isomorphic to $\Gamma_0(E)$ as a Banach module over $\mathrm{C}_0(\Omega)$. Uniqueness follows directly from Lemma 4.8. □

Combining Proposition 4.10 with the preceding Lemmas 4.8 and 4.9, the proof of Theorem 4.6 is straightforward. We skip the details.

Remark 4.11. It is not hard to construct an inverse to the functor of Theorem 4.6. In fact, if Γ is an AM-module over $\mathrm{C}_0(\Omega)$, then we obtain the fibers E_x of a Banach bundle E by setting

$$\begin{aligned} J_x &:= \overline{\mathrm{lin}}\{fs \mid f \in \mathrm{C}_0(\Omega) \text{ with } f(x) = 0 \text{ and } s \in \Gamma\}, \\ E_x &:= \Gamma/J_x, \end{aligned}$$

for $x \in \Omega$, see Section 2 of [DG83] or Section 7 of [Gie98]. Moreover, if $\varphi\colon \Omega \longrightarrow \Omega$ is a homeomorphism and $\mathcal{T} \in \mathscr{L}(\Gamma)$ is a T_φ-homomorphism, then $\mathcal{T} J_x \subseteq J_{\varphi(x)}$ for every $x \in \Omega$ and therefore $\mathcal{T}$ induces a bounded operator $\Phi_x \in \mathscr{L}(E_x, E_{\varphi(x)})$.
With these constructions one can assign a dynamical Banach bundle to a dynamical AM-module $(\Gamma; \mathcal{T})$. We leave the details to the reader (cf. Theorem 2.6 of [DG83]).

4.2. **AL-modules.** The dual concept of AM-spaces in the theory of Banach lattices are so-called AL-spaces (see Section II.8 of [Sch74]). Again we make use of this concept to introduce a certain class of Banach modules.

Definition 4.12. Let Ω be a locally compact space. A Banach module Γ over $\mathrm{C}_0(\Omega)$ is called an *AL-module over* $\mathrm{C}_0(\Omega)$ if Γ_s is an AL-space for each $s \in \Gamma$.

Remark 4.13. A Banach module over $\mathrm{C}_0(\Omega)$ is an AL-module over $\mathrm{C}_0(\Omega)$ if and only if

$$\|f_1 s + f_2 s\| = \|f_1 s\| + \|f_2 s\|$$

for all $f_1, f_2 \in \mathrm{C}_0(\Omega)_+$ and $s \in \Gamma$.

Note that if X is a measure space, then $\mathrm{L}^\infty(X)$ is *-isomorphic to $\mathrm{C}(K)$ for some compact space K. Thus, every Banach module over $\mathrm{L}^\infty(X)$ can be seen as a Banach module over $\mathrm{C}(K)$. In particular, we may speak of *AL-modules over* $\mathrm{L}^\infty(X)$.

Example 4.14. Let E be a measurable Banach bundle over a measure space X. Then $\Gamma^1(E)$ (see Example 3.4) is an AL-module over $\mathrm{L}^\infty(X)$.

Remark 4.15. It is tempting to expect that for a measure space X every AL-module over $\mathrm{L}^\infty(X)$ is already isomorphic to a space $\Gamma^1(E)$ for some measurable Banach bundle E over X. However, we will see below that this is not the case (see Example 5.10).

As in the case of Banach lattices, AM- and AL-modules are dual to each other. To formulate this result we first equip the dual space of a Banach module with a module structure.

Definition 4.16. Let Γ be a Banach module over a commutative C*-algebra A. Then the dual space Γ' equipped with the operation $(f \cdot s')(s) := s'(f \cdot s)$ for $s \in \Gamma$, $s' \in \Gamma'$ and $f \in A$ is the *dual Banach module of* Γ *over* A.

It is straightforward to check that the dual Banach module of a Banach module is in fact a Banach module. We can now make the duality between AM and AL-modules precise using the following result due to Cunnigham (see [Cun67]) though in somewhat different notation. He only treats the compact case. However, using Lemma 3.11 the general result can easily be deduced from this.

Proposition 4.17. *Let* Ω *be a locally compact space. For a Banach module* Γ *over* $\mathrm{C}_0(\Omega)$ *the following assertions hold.*

(i) Γ *is an AM-module if and only if* Γ' *is an AL-module.*

(ii) Γ *is an AL-module if and only if* Γ' *is an AM-module.*

5. Lattice normed modules

5.1. $\mathrm{U}_0(\Omega)$**-normed modules.** As observed in [Cun67], AM-modules admit an additional lattice theoretic structure. For a locally compact space Ω, we write

$$\begin{aligned}
\mathrm{U}(\Omega) &:= \{f \colon \Omega \longrightarrow \mathbb{R} \mid f \text{ upper semicontinuous}\},\\
\mathrm{U}_0(\Omega) &:= \{f \in \mathrm{U}(\Omega) \mid \forall\, \varepsilon > 0\, \exists\, K \subseteq \Omega \text{ compact with } \|s(x)\| \leqslant \varepsilon\, \forall\, x \notin K\},\\
\mathrm{U}_0(\Omega)_+ &:= \{f \in \mathrm{U}_0(\Omega) \mid f \geqslant 0\},
\end{aligned}$$

and introduce the following concept (see Section 6.6 of [HK77] for the compact case).

Definition 5.1. Let Ω be a locally compact space and Γ a Banach module over $\mathrm{C}_0(\Omega)$. A mapping

$$|\cdot|\colon \Gamma \longrightarrow \mathrm{U}_0(\Omega)_+$$

is a $\mathrm{U}_0(\Omega)$-*valued norm* if

(i) $\||s|\| = \|s\|$,

(ii) $|fs| = |f|\cdot|s|$,

(iii) $|s_1+s_2| \leqslant |s_1| + |s_2|$,

for all $s, s_1, s_2 \in \Gamma$ and $f \in \mathrm{C}_0(\Omega)$. A Banach module over $\mathrm{C}_0(\Omega)$ together with a $\mathrm{U}_0(\Omega)$-valued norm is called a $\mathrm{U}_0(\Omega)$-*normed module*.

Example 5.2. Let E be a Banach bundle over a locally compact space Ω. Setting $|s|(x) := \|s(x)\|$ for $x \in \Omega$ and $s \in \Gamma_0(E)$ turns $\Gamma_0(E)$ into a $\mathrm{U}_0(\Omega)$-normed module.

Note that each $\mathrm{U}_0(\Omega)$-normed module is automatically an AM-module over $\mathrm{C}_0(\Omega)$. The converse also holds and is basically due to Cunningham in the compact case (see Lemma 3 and Theorem 2 in [Cun67]).

Proposition 5.3. *Let Ω be a locally compact space. For a Banach module Γ over $\mathrm{C}_0(\Omega)$ the following are equivalent.*

(a) *Γ is an AM-module over A.*

(b) *Γ admits a $\mathrm{U}_0(\Omega)$-valued norm.*

In this case the $\mathrm{U}_0(\Omega)$-valued norm is unique and given by

$$|s|(x) = \inf\{\|fs\| \mid f \in \mathrm{C}_0(\Omega)_+ \textit{ with } f(x) = 1\}$$

for $x \in \Omega$ and $s \in \Gamma$.

Proof. Using Lemma 3.11, existence and the desired formula of the $\mathrm{U}_0(\Omega)$-valued norm can be reduced to the compact case which is treated in Lemma 3 of [Cun67].
For uniqueness, observe that any $\mathrm{U}_0(\Omega)$-valued norm $|\cdot|\colon \Gamma \longrightarrow \mathrm{U}_0(\Omega)_+$ satisfies

$$|s|(x) \leqslant \inf\{\|fs\| \mid f \in \mathrm{C}_0(\Omega)_+ \text{ with } f(x) = 1\}$$

for every $x \in \Omega$ and $s \in \Gamma$. On the other hand, if $x \in \Omega$, $s \in \Gamma$ and $\varepsilon > 0$, we find a neighborhood U of x such that $|s|(y) \leqslant |s|(x) + \varepsilon$ for every $y \in U$ since $|s|$ is upper semicontinuous. Thus, there is $f \in \mathrm{C}_0(\Omega)_+$ with $\|f\| = f(x) = 1$ and

$$\|fs\| = \sup_{y\in\Omega} |fs|(y) = \sup_{y\in\Omega} |f(y)|\cdot|s|(y) \leqslant |s|(x) + \varepsilon$$

which implies the claim. □

Remark 5.4. The representing Banach bundles of AM-modules Γ over $\mathrm{C}_0(\Omega)$ satisfying $|s| \in \mathrm{C}_0(\Omega) \subseteq \mathrm{U}_0(\Omega)$ for every $s \in \Gamma$ are precisely the continuous Banach bundles (see Theorem 15.11 of [Gie98] or pages 47–48 of [DG83] for the compact case; the locally compact case can easily be reduced to this).

We can now state the main theorem of this subsection which shows that the algebraic and lattice theoretic structures of $\mathrm{U}_0(\Omega)$-normed modules are closely related to each other.

Theorem 5.5. *Let Ω be a locally compact space, $\varphi\colon \Omega \longrightarrow \Omega$ a homeomorphism and Γ and Λ $\mathrm{U}_0(\Omega)$-normed modules. For $\mathcal{T} \in \mathscr{L}(\Gamma, \Lambda)$ the following are equivalent.*

(a) *$\mathcal{T}(fs) = T_\varphi f \cdot \mathcal{T}s$ for every $f \in \mathrm{C}_0(\Omega)$ and $s \in \Gamma$.*

(b) *$\mathrm{supp}(\mathcal{T}s) \subseteq \varphi(\mathrm{supp}(s))$ for every $s \in \Gamma$.*

(c) *$|\mathcal{T}s| \leqslant \|\mathcal{T}\| \cdot T_\varphi|s|$ for every $s \in \Gamma$.*

(d) *There is $m > 0$ such that $|\mathcal{T}s| \leqslant m \cdot T_\varphi|s|$ for every $s \in \Gamma$.*

Moreover, if $\Gamma = \Gamma_0(E)$ and $\Lambda = \Gamma_0(F)$ for Banach bundles E and F over Ω, then the properties above are also equivalent to the following assertion.

(e) *There is a morphism Φ over φ with $\mathcal{T} = \mathcal{T}_\Phi$.*

If (e) holds, then the morphism Φ in (e) is unique, $\|\Phi\| = \|\mathcal{T}\|$ and Φ is isometric if and only if $\mathcal{T}$ is isometric.

For the proof we need the following lemma connecting the vector-valued norm with the concept of support introduced in Definition 3.8.

Lemma 5.6. *Let Γ be a $\mathrm{U}_0(\Omega)$-normed module. Then*

$$\mathrm{supp}(s) = \mathrm{supp}(|s|) = \overline{\{x \in \Omega \mid |s|(x) \neq 0\}}$$

for each $s \in \Gamma$.

Proof. Let $x \in \Omega$ with $|s|(x) \neq 0$ and $f \in \mathrm{C}_0(\Omega)$ with $f(x) \neq 0$. Then $|fs|(x) = |f|(x)|s|(x) \neq 0$ and therefore $\|fs\| \neq 0$.
Conversely, let $x \in \mathrm{supp}(s)$. Assume there is an open neighborhood U of x such that $|s|(y) = 0$ for every $y \in U$. We then find a positive function $f \in \mathrm{C}_0(\Omega)$ with support in U and $f(x) = 1$. But then $|fs| = |f||s| = 0$ and therefore $fs = 0$ which contradicts $x \in \mathrm{supp}(s)$. □

Proof (of Theorem 5.5)**.** The equivalence of (a) and (b) holds by Theorem 3.10. Now assume that (a) and (b) hold and there is $s \in \Gamma$ such that $|\mathcal{T}s| \not\leqslant \|\mathcal{T}\| \cdot T_\varphi|s|$. We then find $x \in \Omega$ with $\|\mathcal{T}\| \cdot |s|(x) < |\mathcal{T}s|(\varphi(x))$. Since $|s|$ is upper semi-continuous, we find an open neighborhood V of x such that $\|\mathcal{T}\| \cdot |s|(z) < |\mathcal{T}s|(\varphi(x))$ for all $z \in V$. Now choose a compact neighborhood W of x contained in V. Since $x \in \mathrm{supp}(s)$ by Lemma 5.6 and (b), we find $y \in W$ with $s(y) \neq 0$.
Now take a positive function $f \in \mathrm{C_c}(\Omega)$ supported in V with $0 \leqslant f \leqslant \mathbb{1}$ and $f(z) = 1$ for every $z \in W$. Then $\tilde{s} := fs \neq 0$ and

$$\begin{aligned}\|\mathcal{T}\| \cdot \|\tilde{s}\| &= \sup_{z \in V} \|\mathcal{T}\| \cdot f(z) \cdot |s|(z) \leqslant |\mathcal{T}s|(\varphi(x)) = (T_\varphi f)(\varphi(x)) \cdot |\mathcal{T}s|(\varphi(x)) \\ &= |\mathcal{T}(fs)|(\varphi(x)) \leqslant \|\mathcal{T}\tilde{s}\|,\end{aligned}$$

contradicting the definition of $\|\mathcal{T}\|$.
The implications "(c) $\Rightarrow$ (d)" and "(d) $\Rightarrow$ (b)" are obvious and the rest of the theorem follows from Lemma 4.8. □

Remark 5.7. In view of Proposition 5.3 and Theorem 5.5, the assignments of Theorem 4.6 also define an essentially surjective and fully faithful functor from the category of dynamical Banach bundles over a topological dynamical system $(\Omega;\varphi)$ to the category having as objects pairs of $\mathrm{U}_0(\Omega)$-normed modules and monoid representations of "dominated operators" (in the sense of Theorem 5.5 (c)) and as morphisms operators $V \in \mathscr{L}(\Gamma,\Lambda)$ between $\mathrm{U}_0(\Omega)$-normed modules such that there is an $m > 0$ with $|Vs| \leqslant m \cdot |s|$ for all $s \in \Gamma$.

5.2. $\mathrm{L}^1(X)$**-normed modules.** AL-modules also admit a vector-valued norm.

Definition 5.8. Let A be a commutative C*-algebra and Γ a Banach module over A. A mapping

$$|\cdot|\colon \Gamma \longrightarrow A'_+$$

is an *A'-valued norm* if

(i) $\||s|\| = \|s\|$,

(ii) $|fs| = |f| \cdot |s|$,

(iii) $|s_1 + s_2| \leqslant |s_1| + |s_2|$,

for all $s, s_1, s_2 \in \Gamma$ and $f \in A$. A Banach module over A together with a A'-valued norm is called a *A'-normed module*.

Again the main part of the following result is due to Cunningham in the compact case (see Theorem 4 of [Cun67]). We give a new proof in the general case and also provide an explicit formula for the vector-valued norm.

Proposition 5.9. *Let Ω be a locally compact space. For a Banach module Γ over $\mathrm{C}_0(\Omega)$ the following are equivalent.*

(a) *Γ is an AL-module over $\mathrm{C}_0(\Omega)$.*

(b) *Γ admits a $\mathrm{C}_0(\Omega)'$-valued norm.*

In this case, the $\mathrm{C}_0(\Omega)'$-valued norm is unique and given by $|s|(f) := \|fs\|$ for all $s \in \Gamma$ and $f \in \mathrm{C}_0(\Omega)_+$.

Proof. It is clear that (b) implies (a) since $\mathrm{C}_0(\Omega)'$ is an AL-space by Proposition 9.1 of [Sch74].
If (a) holds, we define $|s|(f) = \|fs\|$ for all $s \in \Gamma$ and $f \in \mathrm{C}_0(\Omega)_+$. For every $s \in \Gamma$ the map $|s|\colon \mathrm{C}_0(\Omega)_+ \to \mathbb{R}_{\geqslant 0}$ is additive and positively homogeneous and therefore has a unique positive extension $|s| \in A'$ by Lemma 1.3.3 of [MN91].
Now take an approximate unit $(e_i)_{i\in I}$ for $\mathrm{C}_0(\Omega)$. Then

$$\|s\| = \lim_i \|e_i s\| = \lim_i |s|(e_i) = \||s|\|,$$

where the last equality follows from Proposition 2.1.5 of [Dix77]. It is clear that $|s_1+s_2| \leqslant |s_1|+|s_2|$ for all $s_1, s_2 \in \Gamma$. Finally, let $f \in \mathrm{C}_0(\Omega)$ and $s \in \Gamma$. Then

$$|fs|(g) = \|gfs\| = \||gf|s\| = |s|(|f|g) = (|f| \cdot |s|)(g)$$

for every $g \in \mathrm{C}_0(\Omega)_+$, where the second equality follows from the fact that Γ_s is a Banach lattice (see Proposition 4.1). This shows $|f \cdot s| = |f| \cdot |s|$.

To prove uniqueness, let $|\cdot|$ be any $\mathrm{C}_0(\Omega)'$-valued norm on Γ and let $(e_i)_{i\in I}$ be an approximate unit for $\mathrm{C}_0(\Omega)$. Then

$$\|fs\| = \lim_i |fs|(e_i) = \lim_i |s|(fe_i) = |s|(f)$$

for each $s \in \Gamma$ and $f \in \mathrm{C}_0(\Omega)_+$, showing the claim. □

If $A = \mathrm{L}^\infty(X)$ for some measure space X, Proposition 5.9 yields a vector-valued norm $|\cdot|\colon \Gamma \longrightarrow \mathrm{L}^\infty(X)'_+$. On the other hand, if E is a measurable Banach bundle over X, then the mapping

$$|\cdot|\colon \Gamma^1(E) \longrightarrow \mathrm{L}^1(X)_+, \quad s \mapsto \|s(\cdot)\|$$

satisfies properties (i) – (iii) of Definition 5.8 and since $\mathrm{L}^1(X)$ embeds canonically (as a lattice ideal and as a Banach module over $\mathrm{L}^\infty(X)$) into $\mathrm{L}^\infty(X)'$, this already defines the unique $\mathrm{L}^\infty(X)'$-valued norm. In particular, an AL-module over $\mathrm{L}^\infty(X)$ can only be isometrically isomorphic to $\Gamma^1(E)$ for some measurable Banach bundle E over X if the $\mathrm{L}^\infty(X)'$-valued norm takes values in (the canonical image of) $\mathrm{L}^1(X)$. This is not always the case as the following example shows.

Example 5.10. Let X be any measure space and consider $\Gamma := \mathrm{L}^\infty(X)'$ as a Banach module over $\mathrm{L}^\infty(X)$. Then Γ is an AL-module over $\mathrm{L}^\infty(X)$ by Proposition 4.17 since $\mathrm{L}^1(X)$ is an AL-module over $\mathrm{L}^\infty(X)$. The usual modulus $|\cdot|\colon \mathrm{L}^\infty(X)' \to \mathrm{L}^\infty(X)'$ is given by

$$|s|(f) = \sup\{|s(g)| \mid 0 \leqslant |g| \leqslant f\}$$

for $f \in \mathrm{L}^\infty(X)_+$ and $s \in \mathrm{L}^\infty(X)'$ (see Corollary 1 to Proposition II.4.2 of [Sch74]). It is easy to see that

$$\sup\{|s(g)| \mid 0 \leqslant |g| \leqslant f\} = \sup\{|s(gf)| \mid 0 \leqslant |g| \leqslant \mathbb{1}\} = \|fs\|$$

for $f \in \mathrm{L}^\infty(X)_+$ and $s \in \mathrm{L}^\infty(X)'$ and therefore $|\cdot|$ is the $\mathrm{L}^\infty(X)'$-valued norm. If $\mathrm{L}^1(X)$ is not finite-dimensional, then $\mathrm{L}^1(X)$ is not reflexive (see Corollary 2 of Theorem II.9.9 in [Sch74]). Thus, there are elements $s \in \Gamma$ with $|s| \notin \mathrm{L}^1(X)$ in this case.

Definition 5.11. Let X be a measure space. An $\mathrm{L}^\infty(X)'$-normed module Γ is called an $\mathrm{L}^1(X)$-*normed module* if $|s| \in \mathrm{L}^1(X)$ for every $s \in \Gamma$.

We now state and prove our second main result. Here a measure space X is *separable* if there is a sequence $(A_n)_{n\in\mathbb{N}}$ of measurable subsets of Ω_X such that for every $B \in \Sigma_X$ and every $\varepsilon > 0$ there is an $n \in \mathbb{N}$ with $\mu_X(A_n \Delta B) < \varepsilon$.

Theorem 5.12. *Let G be a group, $S \subseteq G$ be a submonoid and $(X;\varphi)$ a measure preserving G-dynamical system with X separable. Then the assignments*

$$(E;\Phi) \mapsto (\Gamma^1(E);\mathcal{T}_\Phi)$$
$$\Theta \mapsto V_\Theta$$

define an essentially surjective, fully faithful functor from the category of S-dynamical separable measurable Banach bundles over $(X;\varphi)$ to the category of S-dynamical separable $\mathrm{L}^1(X)$-normed modules over $(\mathrm{L}^\infty(X);T_\varphi)$.

We start by showing that separable Banach bundles over separable measure spaces in fact induce separable spaces of sections.

Proposition 5.13. *Let E be a separable measurable Banach bundle over a separable measure space X. Then $\Gamma^1(E)$ is separable.*

The proof of the following lemma is based on the proof of Proposition 4.4 of [FD88] (see also Lemma A.3.5 of [ADR00] for a similar result).

Lemma 5.14. *Let E be a separable Banach bundle over a measure space X and $(s_n)_{n\in\mathbb{N}}$ in $\mathcal{M}_E$ such that $\mathrm{lin}\{s_n(x) \mid n \in \mathbb{N}\}$ is dense in E_x for almost every $x \in \Omega_X$. Then $\mathrm{lin}\{s_n \mid n\in\mathbb{N}\}$ generates E, i.e., every $s \in \mathcal{M}_E$ is an almost everywhere limit of a sequence in $\mathrm{lin}\{\mathbb{1}_A s_n \mid A \in \Sigma_X, n \in \mathbb{N}\}$.*

Proof. Let $s \in \mathcal{M}_E$, $\varepsilon > 0$ and set

$$A_n := \{x \in \Omega_X \mid \|s(x) - s_n(x)\| < \varepsilon\} \in \Sigma_X$$

for every $n \in \mathbb{N}$. Then

$$\Omega_X \setminus \left(\bigcup_{n\in\mathbb{N}} A_n\right)$$

is a nullset. Therefore, $\|s(x) - \tilde{s}(x)\| < \varepsilon$ for almost every $x \in \Omega_X$ where

$$\tilde{s}(x) = \begin{cases} s_n(x) & x \in A_n \setminus \bigcup_{k=1}^{n-1} A_k, n \in \mathbb{N}, \\ 0 & \text{else.} \end{cases}$$

Since $\tilde{s}$ is a measurable section with respect to the Banach bundle generated by $\mathrm{lin}\{s_n \mid n \in \mathbb{N}\}$ (see Remark 2.12), this shows the claim. □

Lemma 5.15. *Let E be a separable Banach bundle over a measure space X. Then there is a sequence $(s_n)_{n\in\mathbb{N}}$ in $\mathcal{M}_E$ such that*

(i) $\mathrm{lin}\{s_n(x) \mid n \in \mathbb{N}\}$ *is dense in E_x for almost every $x \in \Omega_X$,*

(ii) $\mu_X(\{|s_n| \neq 0\}) < \infty$ *for every $n \in \mathbb{N}$,*

(iii) $|s_n| = \mathbb{1}_{\{|s_n|\neq 0\}}$ *almost everywhere for every $n \in \mathbb{N}$,*

(iv) *for every $x \in \Omega_X$ and $n \in \mathbb{N}$*

$$s_{n+1}(x) = 0 \text{ or } s_{n+1}(x) \notin \mathrm{lin}\{s_1(x), ..., s_n(x)\}.$$

Moreover, for any sequence $(s_n)_{n\in\mathbb{N}}$ in $\mathcal{M}_E$ with properties (i) and (ii), the set

$$\mathrm{lin}\{\mathbb{1}_A s_n \mid A \in \Sigma_X, n \in \mathbb{N}\} \subseteq \Gamma^1(E)$$

is dense in $\Gamma^1(E)$.

Proof. Let $(s_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{M}_E$ such that $\mathrm{lin}\{s_n(x) \mid n \in \mathbb{N}\}$ is dense in E_x for almost every $x \in \Omega_X$. By replacing s_n by $\tilde{s}_n$ defined as

$$\tilde{s}_n(x) := \begin{cases} \frac{1}{\|s_n(x)\|} s_n(x) & s_n(x) \neq 0, \\ 0 & s_n(x) = 0, \end{cases}$$

for every $n \in \mathbb{N}$ we may assume that (i) and (iii) hold. Now pick a sequence $(A_n)_{n\in\mathbb{N}}$ of measurable subsets of Ω_X of finite measure such that

$$\Omega_X = \bigcup_{n\in\mathbb{N}} A_n.$$

Then $\mu_X(\{|\mathbb{1}_{A_m} s_n| \neq 0\}) < \infty$ for all $m, n \in \mathbb{N}$. Replacing $(s_n)_{n\in\mathbb{N}}$ once again, we may assume that properties (i)–(iii) are fulfilled.
We define a new sequence $(\tilde{s}_n)_{n\in\mathbb{N}}$ in $\mathcal{M}_E$ by the following procedure.
First set $\tilde{s}_1 := s_1$. Now assume that $\tilde{s}_n$ is defined for some $n \in \mathbb{N}$. Let Q be a countable dense subset of $\mathbb{K}$. For each $q = (q_1, ..., q_n) \in Q^n$ define

$$f_q := \left| s_k - \sum_{k=1}^{n} q_k \tilde{s}_k \right|.$$

Then f_q is measurable for each $q \in Q^n$ and therefore $f\colon \Omega_X \longrightarrow \mathbb{R}$ defined by $f(x) := \inf_{q\in\mathbb{Q}^n} f_q(x)$ for $x \in \Omega_X$ is also measurable. Note that for $x \in \Omega_X$

$$s_{n+1}(x) \in \mathrm{lin}\{\tilde{s}_1(x), ..., \tilde{s}_n(x)\} \text{ if and only if } f(x) = 0.$$

Therefore, the set $B := \{x \in \Omega \mid s_{n+1}(x) \in \mathrm{lin}\{s_1(x), ..., s_n(x)\}\}$ is measurable. We now set $\tilde{s}_{n+1} := \mathbb{1}_B s_{n+1} \in \mathcal{M}_E$.
Clearly, $\mathrm{lin}\{s_1(x), ..., s_n(x)\} = \mathrm{lin}\{\tilde{s}_1(x), ..., \tilde{s}_n(x)\}$ for all $x \in \Omega_X$ and $n \in \mathbb{N}$ and therefore $(\tilde{s}_n)_{n\in\mathbb{N}}$ has properties (i) – (iv). This shows the existence of a sequence with the desired properties (i) – (iv).
Now assume that $(s_n)_{n\in\mathbb{N}}$ is a sequence $\mathcal{M}_E$ satisfying (i) and (ii) and let $s \in \mathcal{M}_E$ with $\int |s| \,\mathrm{d}\mu_X < \infty$. By Lemma 5.14 and Lemma 4.3 of [FD88] we find a sequence $(t_n)_{n\in\mathbb{N}}$ in

$$M := \mathrm{lin}\{\mathbb{1}_A s_n \mid A \in \Sigma_X, n \in \mathbb{N}\} \subseteq \mathcal{M}_E$$

such that $\lim_{n\to\infty} t_n = s$ almost everywhere and $|t_n| \leqslant |s|$ almost everywhere for all $n \in \mathbb{N}$. By Lebesgue's theorem we therefore obtain that the canonical image of M in $\Gamma^1(E)$ is dense in $\Gamma^1(E)$. □

Proof (of Proposition 5.13)**.** Using the separability of X, we pick a sequence $(A_n)_{n\in\mathbb{N}}$ of measurable subsets of Ω_X such that for every $B \in \Sigma_X$ and every $\varepsilon > 0$ there is $n \in \mathbb{N}$ with $\mu_X(A_n \Delta B) < \varepsilon$. Moreover, take a sequence $(s_n)_{n\in\mathbb{N}}$ as in Lemma 5.15. For each $n \in \mathbb{N}$ and every $A \in \Sigma_X$ we then find an $m \in \mathbb{N}$ with

$$\|\mathbb{1}_A s_n - \mathbb{1}_{A_m} s_n\| \leqslant \mu(A\Delta A_m) < \varepsilon.$$

This implies that $\{\mathbb{1}_{A_m} s_n \mid n, m \in \mathbb{N}\}$ is total in $\Gamma^1(E)$. □

The following result characterizes weighted Koopman operators induced by measurable dynamical Banach bundles similarly to the topological setting (cf. Theorem 5.5).

Theorem 5.16. *Let X be a measure space, $\varphi\colon X \longrightarrow X$ an automorphism and Γ, Λ $\mathrm{L}^1(X)$-normed modules. For an operator $\mathcal{T} \in \mathscr{L}(\Gamma, \Lambda)$ the following are equivalent.*

(a) *$\mathcal{T}(fs) = T_\varphi f \cdot \mathcal{T}s$ for all $f \in \mathrm{L}^\infty(X)$ and every $s \in \Gamma$.*

(b) *$|\mathcal{T}s| \leqslant \|\mathcal{T}\| \cdot T_\varphi |s|$ for every $s \in \Gamma$.*

(c) *There is an $m > 0$ such that $|\mathcal{T}s| \leqslant m \cdot T_\varphi |s|$ for every $s \in \Gamma$.*

Moreover, if $\Gamma = \Gamma^1(E)$ and $\Lambda = \Gamma^1(F)$ for Banach bundles E and F over X with E separable, then the above are also equivalent to the following assertion.

(d) *There is a morphism $\Phi\colon E \longrightarrow F$ over φ such that $\mathcal{T} = \mathcal{T}_\Phi$.*

If (d) holds, then the morphism Φ in (d) is unique,

$$|\Phi|\colon \Omega_X \longrightarrow [0,\infty), \quad x \mapsto \|\Phi_x\|$$

defines an element of $\mathrm{L}^\infty(X)$ and

- *$\sup\{|\mathcal{T}_\Phi s| \mid s \in \Gamma^\infty(E) \text{ with } |s| \leqslant \mathbb{1}\} = T_\varphi |\Phi| \in \mathrm{L}^\infty(X)$,*
- *$\|\Phi\| = \|\mathcal{T}_\Phi\|_{\Gamma^\infty(E)} = \|\mathcal{T}\|_{\Gamma^1(E)}$,*
- *Φ is an isometry if and only $\mathcal{T} \in \mathscr{L}(\Gamma^1(E), \Gamma^1(F))$ is an isometry.*

Proof. Assume that (a) is valid and take $s \in \Gamma$. For each $f \in \mathrm{L}^\infty(X)$ with $f \geqslant 0$

$$\begin{aligned}\langle |\mathcal{T}s|, f\rangle &= \|f\mathcal{T}s\| = \|\mathcal{T}((T_{\varphi^{-1}} f)\cdot s)\|\\ &\leqslant \|\mathcal{T}\| \cdot \|T_{\varphi^{-1}} f \cdot s\| = \|\mathcal{T}\| \cdot \langle |s|, T_{\varphi^{-1}} f\rangle = \langle \|\mathcal{T}\| \cdot T_\varphi |s|, f\rangle\end{aligned}$$

since φ is measure-preserving. Thus, $|\mathcal{T}s| \leqslant \|\mathcal{T}\| \cdot T_\varphi |s|$.

Conversely, assume that (b) holds. Since X is σ-finite, we find measurable and pairwise disjoint sets $A_n \in \Sigma_X$ with finite measure for $n \in \mathbb{N}$ such that

$$\Omega_X = \bigcup_{n\in\mathbb{N}} A_n.$$

For fixed $n \in \mathbb{N}$ consider the submodules

$$\begin{aligned}\Gamma_n &:= \{s \in \Gamma \mid |s| \in \mathrm{L}^\infty(X) \cdot \mathbb{1}_{A_n}\} \subseteq \Gamma,\\ \Lambda_n &:= \{s \in \Lambda \mid |s| \in \mathrm{L}^\infty(X) \cdot \mathbb{1}_{\varphi(A_n)}\} \subseteq \Lambda.\end{aligned}$$

We define $\|s\|_\infty := \||s|\|_{\mathrm{L}^\infty(X)}$ for $s \in \Gamma_n$ and $s \in \Lambda_n$, respectively. If $(s_m)_{n\in\mathbb{N}}$ is a Cauchy sequence in Γ_n with respect to the norm $\|\cdot\|_\infty$, then by completeness of Γ there is $s \in \Gamma$ such that $\lim_{n\to\infty} s_m = s$ in Γ. Using that there is a subsequence $(s_{n_k})_{k\in\mathbb{N}}$ of $(s_m)_{m\in\mathbb{N}}$ such that $|s_{m_k} - s| \to 0$ almost everywhere, it follows that $s \in \Gamma_n$ and $\lim_{m\to\infty} s_m = s$ with respect to $\|\cdot\|_\infty$.

Thus, Γ_n and likewise Λ_n is a Banach module over $\mathrm{L}^\infty(X)$. Moreover, $\mathcal{T}|_{\Gamma_n} \in \mathscr{L}(\Gamma_n, \Lambda_n)$.

Choose a *-isomorphism $V \in \mathscr{L}(\mathrm{L}^\infty(X), \mathrm{C}(K))$ for some compact space K. We then consider Γ_n and Λ_n as Banach modules over $\mathrm{C}(K)$ via V^{-1} and see that the mappings

$$\begin{aligned}\Gamma_n &\longrightarrow \mathrm{C}(K), \quad s \mapsto V|s|,\\ \Lambda_n &\longrightarrow \mathrm{C}(K), \quad s \mapsto V|s|\end{aligned}$$

turn Γ_n and Λ_n into $\mathrm{U}(K)$-normed modules. Applying Theorem 5.5 then shows that $\mathcal{T}(fs) = (T_\varphi f) \cdot \mathcal{T}s$ for all $f \in \mathrm{L}^\infty(X)$ and $s \in \Gamma_n$.
Take $f \in \mathrm{L}^\infty(X)$ and $s \in \Gamma$ with $|s| \in \mathbb{1}_{A_n}\mathrm{L}^1(X)$. Then $s = \lim_{n\to\infty} \mathbb{1}_{\{|s|\leqslant n\}}s$ in Γ and therefore

$$\mathcal{T}(fs) = \lim_{n\to\infty} \mathcal{T}(f\mathbb{1}_{\{|s|\leqslant n\}}s) = \lim_{n\to\infty} (T_\varphi f) \cdot \mathcal{T}(\mathbb{1}_{\{|s|\leqslant n\}}s) = (T_\varphi f) \cdot \mathcal{T}s.$$

Finally,

$$\begin{aligned}\mathcal{T}(fs) &= \lim_{N\to\infty} \mathcal{T}\left(f\sum_{n=1}^{N} \mathbb{1}_{A_n}s\right) = \lim_{N\to\infty}\sum_{n=1}^{N} \mathcal{T}(f\mathbb{1}_{A_n}s) \\ &= \lim_{N\to\infty}\sum_{n=1}^{N} T_\varphi f \cdot \mathcal{T}\mathbb{1}_{A_n}s = T_\varphi f \cdot \mathcal{T}s\end{aligned}$$

for every $f \in \mathrm{L}^\infty(X)$ and $s \in \Gamma$.

Now assume that $\Gamma = \Gamma^1(E)$ and $\Lambda = \Gamma^1(F)$ for measurable Banach bundles E and F over X with E separable. Let $(s_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{M}_E$ as in Lemma 5.15 and set

$$H_x := \mathrm{lin}\{s_k(x) \mid k \in \mathbb{N}\}$$

for every $x \in \Omega_X$.
Let $\mathcal{T}$ be a T_φ-homomorphism. We choose a representative $t_n \in \mathcal{M}_F$ of $\mathcal{T}s_n \in \Gamma^1(F)$ for each $n \in \mathbb{N}$. Since

$$\|t_n(\varphi(x))\| \leqslant \|\mathcal{T}\| \cdot \|s_n(x)\|$$

for almost every $x \in \Omega_X$ and $n \in \mathbb{N}$ by (b), we find a unique linear mapping $\Phi_x \in \mathscr{L}(H_x, E_{\varphi(x)})$ such that $\Phi_x s_n(x) = (t_n)(\varphi(x))$ for every $n \in \mathbb{N}$ and almost every $x \in \Omega_X$. For almost every $x \in \Omega_X$ the map Φ_x is bounded with $\|\Phi_x\| \leqslant \|\mathcal{T}\|$ and has a unique extension to a bounded operator on E_x which we also denote by Φ_x. We set $\Phi_x := 0 \in \mathscr{L}(E_x)$ for the remaining points $x \in \Omega_X$ and obtain a mapping

$$\Phi\colon E \longrightarrow F, \quad v \mapsto \Phi_{p_E(v)}v.$$

Since $\Phi \circ (\mathbb{1}_A \cdot s_n) = (\mathbb{1}_{\varphi(A)} \cdot t_n) \circ \varphi$ almost everywhere for every $n \in \mathbb{N}$ and every set $A \in \Sigma_X$, we can apply Lemma 5.14 to see that for each $s \in \mathcal{M}_E$ there is a $t \in \mathcal{M}_F$ with $\Phi \circ s = t \circ \varphi$ almost everywhere. This shows that Φ is a morphism of measurable Banach bundles over φ. Moreover, $\mathcal{T}_\Phi s_n = \mathcal{T}s_n$ and, since $\{s_n \mid n \in \mathbb{N}\}$ defines a total subset of $\Gamma^1(E)$, we obtain $\mathcal{T} = \mathcal{T}_\Phi$. Thus (a), (b) and (c) imply (d). The converse implication is obvious.

Now let $\Phi\colon E \longrightarrow F$ be a morphism over φ. Using standard arguments, we find a sequence $(\tilde{s}_n)_{n\in\mathbb{N}}$ in $\mathcal{M}_E$ such that

- $|\tilde{s}_n| \leqslant \mathbb{1}$ almost everywhere for every $n \in \mathbb{N}$,
- $\mu_X(\{|\tilde{s}_n| \neq 0\}) < \infty$ for every $n \in \mathbb{N}$,
- $\{\tilde{s}_n(x) \mid n \in \mathbb{N}\}$ is dense in the unit ball B_{E_x} of E_x for almost every $x \in \Omega_X$.

If $\tilde{\Phi}$ is a premorphism representative of Φ, then

$$\|\tilde{\Phi}_{E_x}\| = \sup_{n\in\mathbb{N}} \|\tilde{\Phi}_{E_x}\tilde{s}_n(x)\|$$

for almost every $x \in \Omega_X$. Thus, $\Omega_X \longrightarrow \mathbb{R}$, $x \mapsto \|\tilde{\Phi}_{E_x}\|$ is measurable and $|\Phi|$ defines an element of $\mathrm{L}^\infty(X)$ of norm $\|\Phi\|$.
Clearly,

$$|\mathcal{T}_\Phi| := \sup\{|\mathcal{T}_\Phi s| \mid s \in \Gamma^\infty(E) \text{ with } |s| \leqslant \mathbb{1}\} \leqslant T_\varphi|\Phi|$$

in $\mathrm{L}^\infty(X)$ (note that the supremum on the left hand side exists since the Banach lattice $\mathrm{L}^\infty(X)$ is order complete, see Example 5 on page 106 and the Corollary of Proposition II.7.7 of [Sch74]). On the other hand,

$$\begin{aligned}T_\varphi|\Phi|(x) = \|\Phi_{\varphi^{-1}(x)}\| = \sup_{n\in\mathbb{N}} \|\Phi_{\varphi^{-1}(x)}\tilde{s}_n(\varphi^{-1}(x))\| = \sup_{n\in\mathbb{N}} \|(\mathcal{T}_\Phi \tilde{s}_n)(x)\| \\ \leqslant |\mathcal{T}_\Phi|(x)\end{aligned}$$

for almost every $x \in \Omega_X$, showing that $T_\varphi|\Phi| = |\mathcal{T}_\Phi|$. Moreover,

$$\begin{aligned}\|\Phi\| &= \operatorname{ess\,sup}_{x\in\Omega_X} \sup_{n\in\mathbb{N}} \|(\mathcal{T}_\Phi \tilde{s}_n)(x)\| = \sup_{n\in\mathbb{N}} \operatorname{ess\,sup}_{x\in\Omega_X} \|(\mathcal{T}_\Phi \tilde{s}_n)(x)\| \\ &= \sup_{n\in\mathbb{N}} \|\mathcal{T}_\Phi \tilde{s}_n\|_\infty \leqslant \|\mathcal{T}_\Phi\|_{\Gamma^\infty(E)},\end{aligned}$$

and $\|\mathcal{T}_\Phi\|_{\Gamma^\infty(E)} \leqslant \|\Phi\|$ is clear, hence $\|\mathcal{T}_\Phi\|_{\Gamma^\infty(E)} = \|\Phi\| = \||\mathcal{T}_\Phi|\|_{\mathrm{L}^\infty(X)}$.
Now pick $s \in \Gamma^\infty(E)$ with $|s| \leqslant \mathbb{1}$. For every measurable set $A \in \Sigma_X$ with finite measure

$$\mathbb{1}_A|\mathcal{T}_\Phi s| = |\mathcal{T}_\Phi(T_\varphi^{-1}\mathbb{1}_A \cdot s)| \leqslant \|\mathcal{T}_\Phi\|_{\Gamma^1(E)} \cdot T_\varphi|(T_\varphi^{-1}\mathbb{1}_A \cdot s)| \leqslant \|\mathcal{T}_\Phi\|_{\Gamma^1(E)} \cdot \mathbb{1}_A$$

by (b). Since X is σ-finite, we obtain $\||\mathcal{T}_\Phi|\|_{\mathrm{L}^\infty(X)} \leqslant \|\mathcal{T}_\Phi\|_{\Gamma^1(E)}$ and the inequality $\|\mathcal{T}_\Phi\|_{\Gamma^1(E)} \leqslant \|\Phi\|$ is obvious. Therefore,

$$\|\Phi\| = \|\mathcal{T}_\Phi\|_{\Gamma^\infty(E)} = \|\mathcal{T}_\Phi\|_{\Gamma^1(E)}$$

and, since the difference of premorphisms over φ is again a premorphism over φ, this equality also proves the uniqueness of Φ in (d).
If Φ is an isometry, then clearly $\mathcal{T}_\Phi$ is an isometry. Assume conversely that $\mathcal{T}_\Phi$ is an isometry and pick a representative $\tilde{\Phi}$ of Φ. We already know that $\tilde{\Phi}_{E_x}$ is a contraction for almost every $x \in \Omega_X$.
Assume that there is a set $A \in \Sigma_X$ with positive measure such that $\tilde{\Phi}|_{E_x}$ is not an isometry for every $x \in A$. We then find an $n \in \mathbb{N}$ and a set $B \in \Sigma_X$ with positive measure such that $\|\Phi_x \tilde{s}_n(x)\| < \|\tilde{s}_n(x)\|$ for every $x \in B$. This implies

$$\|\mathcal{T}_\Phi \tilde{s}_n\| = \int_X \|\Phi_x \tilde{s}_n(x)\| \,\mathrm{d}\mu_X < \int_X \|\tilde{s}_n(x)\| \,\mathrm{d}\mu_X = \|\tilde{s}_n\|,$$

a contradiction. □

Since we have not employed any continuity assumptions on dynamical measurable Banach bundles, we immediately obtain the following consequence of Theorem 5.16.

Corollary 5.17. *Let G be a (discrete) group, $S \subseteq G$ be a submonoid and $(X;\varphi)$ a measure-preserving G-dynamical system. Moreover let E be a separable Banach bundle over X and let $\mathcal{T}\colon S \longrightarrow \mathscr{L}(\Gamma^1(E))$ be a monoid representation such that $(\Gamma^1(E);\mathcal{T})$ is an S-dynamical Banach module over $(\mathrm{L}^\infty(X);T_\varphi)$. Then there is a unique dynamical Banach bundle $(E;\Phi)$ over $(X;\varphi)$ such that $\mathcal{T}_\Phi = \mathcal{T}$.*

Finally, we use a result of Gutmann ([Gut93b]) to represent $\mathrm{L}^1(X)$-normed modules.

Proposition 5.18. *Let X be a measure space and Γ an $\mathrm{L}^1(X)$-normed module. Then the following assertions hold.*

(i) *There is a measurable Banach bundle E over X such that $\Gamma^1(E)$ is isometrically isomorphic to Γ.*

(ii) *If Γ is separable, then there is a separable Banach bundle E over X such that $\Gamma^1(E)$ is isometrically isomorphic to Γ. Moreover, E is unique up to isometric isomorphy.*

Proof. In the real case, 7.1.3 of [Kus00] shows that the space Γ is in particular a Banach–Kantorovich space over $\mathrm{L}^1(X)$ (see Chapter 2 of [Kus00] for this concept) and we find a measurable Banach bundle E over X such that Γ is isometrically isomorphic to $\Gamma^1(E)$ as a lattice normed space by Theorem 3.4.8 of [Gut93b][2]. If we start with a complex $\mathrm{L}^1(X)$-normed module, the proof of this theorem reveals that the constructed Banach bundle E is canonically a Banach bundle of complex Banach spaces and that the isomorphism of Γ and $\Gamma^1(E)$ is $\mathbb{C}$-linear (see Theorem 3.3.4 of [Gut93b] and Theorems 2.1.5 and 2.4.2 of [Gut93a]). In any case, we can apply Theorem 5.16 to see that this isomorphism is an isometric Banach module isomorphism.
Now assume that Γ and therefore $\Gamma^1(E)$ is separable. Let $(s_n)_{n\in\mathbb{N}}$ be dense in $\Gamma^1(E)$ and choose a representative in $\mathcal{M}_E$ for each s_n (which we also denote by s_n). We define a new measurable Banach bundle by setting $F_x := \overline{\{s_n(x) \mid n \in \mathbb{N}\}}$ for every $x \in \Omega_X$ and

$$\mathcal{M}_F := \{s \in \mathcal{M}_E \mid s(x) \in F_x \text{ for every } x \in \Omega_X\}.$$

Then

$$V : \Gamma^1(F) \longrightarrow \Gamma^1(E), \quad s \mapsto s$$

is an isometric module homomorphism. However, since $s_n \in \Gamma^1(F)$ for every $n \in \mathbb{N}$, V is in fact an isometric isomorphism. Clearly, F is separable. Uniqueness up to isometric isomorphy follows immediately from Theorem 5.16. □

Combining Lemma 5.14, Corollary 5.17, Theorem 5.16 and Proposition 5.18 now yields Theorem 5.12.

Remark 5.19. Note that in contrast to the topological setting, the construction of the representing separable measurable Banach bundle is not canonical and involves choices.

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3 Additional Manuscripts

3.1 Uniform enveloping semigroupoids for extensions of topological dynamical systems

UNIFORM ENVELOPING SEMIGROUPOIDS FOR EXTENSIONS OF TOPOLOGICAL DYNAMICAL SYSTEMS

NIKOLAI EDEKO AND HENRIK KREIDLER

Abstract. We study isometric and pseudoisometric extensions of topological dynamical systems and prove two new characterizations for such extensions. Our starting point is that, in the study of extensions $q\colon (K;G) \to (L;G)$ of dynamical systems, the enveloping Ellis semigroup $\mathrm{E}(K;G)$ has certain limitations when the systems are nonminimal. This motivates the concept of *enveloping semigroupoids* as a generalization of enveloping semigroups that is adapted to extensions. We introduce the *uniform enveloping semigroupoid* $\mathcal{E}_{\mathrm{u}}(q)$ of an extension and, under appropriate assumptions, show that an extension q is (pseudo)isometric if and only if $\mathcal{E}_{\mathrm{u}}(q)$ is a compact groupoid. We then prove a Peter-Weyl-type theorem for representations of compact, transitive groupoids on Banach bundles and use this to derive an operator theoretic characterization for pseudoisometric extensions.

Given a topological dynamical system $(K;\varphi)$ consisting of a compact space K and a continuous map $\varphi\colon K \to K$, its enveloping Ellis semigroup $\mathrm{E}(K;\varphi)$ introduced by Ellis in [Ell60] as the pointwise closure

$$\mathrm{E}(K;\varphi) := \overline{\{\varphi^n \mid n \in \mathbb{N}\}} \subseteq K^K$$

is an important tool in topological dynamics. It is a compact, right-topological semigroup that allows to study a dynamical system in terms of topological and algebraic properties of its enveloping semigroup. In particular, it allows to apply the theory of compact, right-topological semigroups to topological dynamics. But apart from single systems, the Ellis semigroup also allows to understand extensions

$$q\colon (K;\varphi) \to (L;\psi)$$

of systems by taking elements $\vartheta \in \mathrm{E}(K;\varphi)$ and restricting them to fibers $K_l := q^{-1}(l)$ of q. These restrictions $\vartheta_l := \vartheta|_{K_l}$ capture properties of the system $(K;\varphi)$ relative to $(L;\psi)$ and the resulting *fiber semigroups*

$$\mathrm{E}_l(K;\varphi) := \{\vartheta_l \mid \vartheta \in \mathrm{E}(K;\varphi)\colon \vartheta(K_l) \subseteq K_l\}$$

acting on the fibers of q serve as a "relativized" version of the Ellis semigroup and yield many results about extensions of minimal systems, see, e.g., [Bro79, Section 3.14].

However, for systems that are not pointwise recurrent (a property guaranteed by minimality), this approach of "relativizing" the Ellis semigroup has not proven very fruitful since

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a fiber semigroup $\mathrm{E}_l(K;\varphi)$ is nonempty if and only if $l \in L$ is a recurrent point. This means that there is an inherent dependence on recurrence in the approach of considering the semigroup $\{\varphi^n \mid n \in \mathbb{N}\}$, forming the enveloping semigroup $\mathrm{E}(K;\varphi)$, and then considering its restrictions $\mathrm{E}_l(K;\varphi)$ to fibers of q. Yet, as we discuss in Section 1, there are many examples for extensions of nonminimal systems for which properties that can be proved in the minimal setting still hold, suggesting that minimality is inessential for certain purposes. Thus, in order to extend the scope of the fiber semigroup approach, we suggest to work around this built-in dependence on recurrence and introduce the concept of *enveloping semigroupoids* as a generalization of enveloping semigroups. Concretely, we change the above-explained approach by first forming the *semigroupoid* of restrictions

$$\mathcal{S}(q) := \left\{\varphi_l^n \,\middle|\, n \in \mathbb{N}, l \in L\right\},$$

passing to its *uniform enveloping semigroupoid* $\mathcal{E}_\mathrm{u}(\mathcal{S}(q))$. Then we consider the semigroups formed by the elements in $\mathcal{E}_\mathrm{u}(\mathcal{S}(q))$ that act on a given fiber K_l.

Groupoids, generalizations of groups that allow to capture local symmetry, play an important role in, e.g., noncommutative geometry where they have provided a framework for studying operator algebras, index theory, and foliations (see [Con95] or [MS06]). In ergodic theory, Mackey used groupoids for his theory of virtual groups in order "to bring to light and exploit certain apparently far reaching analogies between group theory and ergodic theory" ([Mac66, p. 187 and Section 11]). It is the purpose of this article to show that groupoids have also been implicitly used in the study of extensions of topological dynamical systems and that the systematic analysis of the occurring groupoid structures allows to simplify and generalize results on isometric and equicontinuous extensions by replacing minimality with the considerably weaker condition $\dim \mathrm{fix}(T_\varphi) = 1$ for the Koopman operator $T_\varphi \colon \mathrm{C}(K) \to \mathrm{C}(K),\ f \mapsto f \circ \varphi$.

The importance of isometric and equicontinuous extensions of topological dynamical systems is in particular due to the Furstenberg structure theorem for distal minimal flows which states that any distal minimal flow can be constructed via a *Furstenberg tower* consisting of isometric (equivalently: equicontinuous) extensions. In this article, we are interested in an operator theoretic characterization of these extensions. For (invertible) topological dynamical systems it is known that equicontinuity and the Koopman operator T_φ having discrete spectrum are equivalent. The following consequence of results of A. W. Knapp from [Kna67] gives hope that a similar characterization can be derived for extensions. Note here, that every continuous surjection $q\colon K \to L$ canonically induces a $\mathrm{C}(L)$-module structure on $\mathrm{C}(K)$.

Theorem. *Let $q\colon (K;\varphi) \to (L;\psi)$ be an equicontinuous extension of invertible, minimal, and distal topological dynamical systems. Then the union of all finitely generated closed T_φ-invariant $\mathrm{C}(L)$-submodules of $\mathrm{C}(K)$ is dense in $\mathrm{C}(K)$.*

As one of our main applications of the uniform enveloping semigroupoid, we extend this to a characterization of nonminimal, not necessarily distal systems in the following way. Recall that a module Γ over a commutative unital ring R is *projective* if there is an R-module $\widetilde{\Gamma}$ such that the module $\Gamma \oplus \widetilde{\Gamma}$ is free, i.e., has a basis.

Theorem. *For an open extension* $q\colon (K;\varphi) \to (L;\psi)$ *with* $\dim \operatorname{fix}(T_\psi) = 1$ *the following assertions are equivalent.*

(a) *q is a pseudoisometric extension.*

(b) *The union of all finitely generated and projective closed T_φ-invariant $\mathrm{C}(L)$-submodules of $\mathrm{C}(K)$ is dense in $\mathrm{C}(K)$.*

Note that the openness condition on the extension is automatically fulfilled in the case of minimal and distal systems (see [Bro79, Corollary 3.12.25]). Our key tool for the difficult implication (a) $\implies$ (b) is the following Peter-Weyl-type theorem for compact, transitive groupoids which we prove in Theorem 3.6.

Theorem. *Let T be a continuous representation of a compact transitive groupoid $\mathcal{G}$ on a Banach bundle E over the unit space $\mathcal{G}^{(0)}$. Then the union of all invariant subbundles of constant finite dimension is fiberwise dense in E. If, moreover, $\mathcal{G}$ is abelian, then the union of all invariant subbundles of constant dimension one is fiberwise total in E.*

Apart from this, we also generalize another result of Knapp on the existence of unique relatively invariant measures for equicontinuous extensions (see Theorem 2.7).

Organization of the article. In Section 1, we discuss equicontinuous and related extensions, prove new characterizations for such extensions, and in particular establish a generalized Arzelà-Ascoli theorem for extensions. We motivate and then introduce the uniform enveloping semigroupoid of an extension and characterize its compactness. In Section 2 we then consider Haar systems for the isotropy bundles of compact transitive groupoids and use them to show the existence of relatively invariant measures for open pseudoisometric extensions of topologically ergodic systems. Section 3 is then devoted to a Peter-Weyl-type theorem for representations of compact transitive groupoids on Banach bundles. We apply this to the uniform enveloping groupoids of pseudoisometric extensions to derive the characterization stated above.

Terminology and Notation. All compact spaces are assumed to be Hausdorff though we may occasionally specify the Hausdorff property for emphasis. The neighborhood filter of a point $x \in X$ in a topological space X is denoted by $\mathcal{U}_X(x)$ or simply $\mathcal{U}(x)$ when there is no room for ambiguity. If X is a uniform space, we write $\mathcal{U}_X$ for the uniform structure of X.

At several points in the paper we consider bundles, i.e., continuous surjections $p\colon E \to L$ for some topological *total space* E (usually with some additional structure) to a topological (usually compact) *base space* L. For $l \in L$, we write $E_l := p^{-1}(l)$ for the *fiber over* l of such a bundle, and if $f\colon E \to X$ is a map to some set X, we set $f_l := f|_{E_l}$. Moreover, if $p_1\colon E_1 \to L$ and $p_2\colon E_2 \to L$ are two bundles over the same base space L, we define

$$E_1 \times_{p_1,p_2} E_2 := \{(x,y) \in E_1 \times E_2 \mid p_1(x) = p_2(y)\} \subseteq E_1 \times E_2$$

and equip this set with the subspace topology induced by the product topology on $E_1 \times E_2$. We also write $E_1 \times_L E_2 := E_1 \times_{p_1,p_2} E_2$ if the mappings p_1 and p_2 are clear.

We use the letters S and G for topological semigroups and groups and the letters $\mathcal{S}$ and $\mathcal{G}$ for semigroupoids and groupoids, respectively. By a *topological dynamical system* we mean a triple $(K; S, \varphi)$ consisting of a non-empty compact space K, a topological semigroup S, and a continuous action $\varphi\colon S \times K \to K$ of S on K. If S contains a neutral element 1_S, we require that $\varphi(1_S, \cdot) = \mathrm{id}_K$, so that if S is a group, φ is automatically a group action. For $s \in S$, we denote the map $\varphi(s, \cdot)\colon K \to K$ by φ_s. We omit φ from the notation if there is no room for confusion and if $S \in \{\mathbb{N}, \mathbb{Z}\}$, we abbreviate $(K; S, \varphi)$ by $(K; \varphi)$ and identify φ with the map $\varphi(1, \cdot)\colon K \to K$ that completely determines the action. If $(K; \varphi)$ is specified as invertible, we consider the system as a $\mathbb{Z}$-action, otherwise we assume an $\mathbb{N}$-action.

As usual, a *morphism* $q\colon (K; S, \varphi) \to (L; S, \psi)$ between dynamical systems $(K; S, \varphi)$ and $(L; S, \psi)$ is a continuous mapping $q\colon K \to L$ such that the diagram

$$\begin{tikzcd} K \arrow[r, "\varphi_s"] \arrow[d, "q"'] & K \arrow[d, "q"] \\ L \arrow[r, "\psi_s"'] & L \end{tikzcd}$$

commutes for all $s \in S$. A morphism $q\colon (K; S, \varphi) \to (L; S, \psi)$ is an *extension (of topological dynamical systems)* if the underlying map $q\colon K \to L$ is surjective.

Finally, if K is a compact space, we write $\mathrm{C}(K)$ for the Banach space of all continuous complex-valued functions on K. We identify its dual space $\mathrm{C}(K)$ with the space of all complex regular Borel measures on K and write $\mathrm{P}(K) \subseteq \mathrm{C}(K)'$ for the space of all probability measures in $\mathrm{C}(K)'$. If $\vartheta\colon K \to L$ is a continuous mapping between compact spaces K and L we write $\vartheta_*\mu$ for the pushforward of a measure $\mu \in \mathrm{C}(K)'$, i.e.,

$$\int_L f \,\mathrm{d}\vartheta_*(\mu) = \int_K f \circ \vartheta \,\mathrm{d}\mu \quad \text{for } f \in \mathrm{C}(L).$$

Moreover, we define the *Koopman operator* $T_\vartheta \in \mathscr{L}(\mathrm{C}(L), \mathrm{C}(K))$ of ϑ by $T_\vartheta f := f \circ \vartheta$ for $f \in \mathrm{C}(L)$. For a topological dynamical system $(K; S, \varphi)$, the mapping

$$T_\varphi\colon S \to \mathscr{L}(\mathrm{C}(K)), \quad s \mapsto T_{\varphi_s}$$

is the *Koopman (anti)representation* of $(K; S, \varphi)$.

1. Uniform enveloping semigroupoids

The famous Peter-Weyl theorem (see [Fol15, Section 5.2]) on representations of compact groups has numerous applications among which is in particular the following result in representation theory (see [EFHN15, Theorem 15.14]).

Theorem 1.1. *Let $\pi\colon G \to \mathscr{L}(E)$ be a strongly continuous representation of a compact group G on a Banach space E. Then the union of all finite-dimensional G-invariant subspaces of E is dense in E.*

In topological dynamics, this allows to prove the following characterization of the equicontinuity of a group action.

Theorem 1.2. *Let $(K;G)$ be a topological dynamical system. Then the following assertions are equivalent.*

(a) *$(K;G)$ is equicontinuous.*

(b) *The union of all finite-dimensional G-invariant subspaces is dense in* $\mathrm{C}(K)$.

One approach to this theorem uses the enveloping Ellis semigroup

$$\mathrm{E}(K;S,\varphi) := \overline{\{\varphi_s \mid s \in S\}} \subseteq K^K$$

of a system $(K;S,\varphi)$ and the fact that if $S = G$ is a group, then $(K;G)$ is equicontinuous if and only if $\mathrm{E}(K;G)$ is a compact topological group of continuous functions on K. This allows to use Theorem 1.1 to prove the difficult implication (a) $\implies$ (b) in Theorem 1.2 by using the compact group $\mathrm{E}(K;G)$ and its Koopman representation.

The main goal of this article is to prove a common generalization of Theorem 1.2 and Knapp's above-mentioned result to extensions of nonminimal dynamical systems. To this end, we develop the notion of uniform enveloping semigroupoids as an alternative to the enveloping Ellis semigroup and prove a generalization of Theorem 1.1 to representations of compact, transitive groupoids.

Equicontinuous extensions. We start by recalling the following notions of extensions as well as the relations between them.

Definition 1.3. Let $q\colon (K;S) \to (L;S)$ be an extension of topological dynamical systems. Then q is called

(i) *weakly equicontinuous* or *stable* if for each $l \in L$ and each entourage $U \in \mathcal{U}_K$ there is an entourage $V \in \mathcal{U}_K$ such that one has $(sx_1, sx_2) \in U$ for all $s \in S$ and $(x_1, x_2) \in V$ with $x_1, x_2 \in K_l$.

(ii) *equicontinuous* if for each entourage $U \in \mathcal{U}_K$ there is an entourage $V \in \mathcal{U}_K$ such that for each $l \in L$ one has $(sx_1, sx_2) \in U$ for all $s \in S$ and $(x_1, x_2) \in V$ with $x_1, x_2 \in K_l$.

(iii) *pseudoisometric* if there is a set P of continuous mappings $p\colon K \times_L K \to [0,\infty)$ such that
   - $p_l = p|_{K_l \times K_l}$ is a pseudometric on K_l for every $l \in L$,
   - the pseudometrics p_l for $p \in P$ generate the topology of K_l for every $l \in L$,
   - $p(sx, sy) = p(x, y)$ for all $s \in S$ and $x, y \in K$ with $q(x) = q(y)$.

(iv) *isometric* if it is pseudoisometric and the set P can be chosen to consist of a single map which is (necessarily) a metric on each fiber.

Remark 1.4. With Proposition 1.5 below we obtain that (iv) $\implies$ (iii) $\implies$ (ii) $\implies$ (i). If $(K;G)$ and $(L;G)$ are minimal group actions, q is equicontinuous if and only if it is pseudoisometric if and only if it is weakly equicontinuous and open, see [dVr93, Corollary 5.10] and [Bro79, Theorem 3.13.17]. Moreover, all of the notions in Definition 1.3 differ in general, even for extensions of invertible systems: For (iii) and (iv) this is obvious, for (ii) and (iii) see Example 1.27 below, and for the relation between and (i) and (ii) we refer to [Aus13].

Proposition 1.5. *Let $q\colon (K;S) \to (L;S)$ be a pseudoisometric extension of topological dynamical systems. Then q is equicontinuous.*

Proof. Pick a set P as in Definition 1.3 (iii) and take $U \in \mathcal{U}_K$. For each finite subset $F \subseteq P$ and $\varepsilon > 0$, set

$$U_{F,\varepsilon} := \{(x,y) \in K \times_L K \mid \forall\, p \in F\colon p(x,y) < \varepsilon\}$$

and note that

$$\bigcap_{\substack{F \subseteq P \text{ finite} \\ \varepsilon > 0}} U_{F,\varepsilon} = \Delta_K.$$

We claim that for every $U \in \mathcal{U}_K$, there are a finite set $F \subseteq P$ and an $\varepsilon > 0$ such that $U_{F,\varepsilon} \subseteq U$ which would yield the claim since $U_{F,\varepsilon}$ is S-invariant. In order to prove the claim, first recall that $\mathcal{U}_K = \mathcal{U}_{K\times K}(\Delta_K)$ is just the neighborhood filter of the diagonal. The claim then follows from the fact that if $(M_\alpha)_{\alpha\in A}$ is a decreasing family of sets in a compact space X and U is an open neighborhood of $\bigcap_{\alpha\in A} \overline{M_\alpha}$, then there is an $\alpha_0 \in A$ such that $M_{\alpha_0} \subseteq U$ (use the finite intersection property). □

The types of extensions in Definition 1.3 are well-understood for extensions $q\colon (K;G) \to (L;G)$ of minimal group actions and, in particular, enjoy several pleasant properties such as the existence of relatively invariant measures for equicontinuous extensions (see [Kna67, Proposition 5.5] or [Gla75, Corollary 3.7]) or Knapp's above-mentioned result. The key for obtaining these results are the *Ellis fiber semigroups*

$$\mathrm{E}_l(K;S) := \{\vartheta_l \mid \vartheta \in \mathrm{E}(K;S)\colon \vartheta(K_l) \subseteq \vartheta(K_l)\}$$

derived from the Ellis semigroup of a system $(K;S)$. Unfortunately, this approach to equicontinuous extensions breaks down for systems that are not pointwise recurrent since $\mathrm{E}_l(K;S)$ is nonempty if and only if $l \in L$ is recurrent. Yet, the following examples illustrate that there are many nonminimal extensions for which the union of all invariant $\mathrm{C}(L)$-submodules of $\mathrm{C}(K)$ of constant finite dimesion is still dense, even if almost all Ellis fiber semigroups are empty.

Examples 1.6. One readily verifies for all the following equicontinuous extensions $q\colon (K;S) \to (L;S)$ that the $\mathrm{C}(L)$-submodules of $\mathrm{C}(K)$ of constant finite dimension are dense.

1) Let $(K;G)$ be a nonminimal equicontinuous system and consider the factor map $q\colon (K;G) \to (\mathrm{pt};G)$ onto a point. Then the conclusion of Theorem 1.2 may be reformulated by saying that the invariant $\mathrm{C}(\mathrm{pt})$-submodules of constant finite dimension of $\mathrm{C}(K)$ are dense.

2) Let $(K;\varphi)$ be a nonminimal invertible equicontinuous system. Then the Ellis semigroup $\mathrm{E}(K;\varphi)$ is a compact topological group consisting of continuous maps and the quotient $L := K/\mathrm{E}(K;\varphi)$ is a compact Hausdorff which yields a factor map $p\colon (K;\varphi) \to (L;\mathrm{id}_L)$. If P is a family of pseudometrics generating the topology of K, define for $p \in P$
$$p'\colon K \times_L K \to [0,\infty), \quad p'(x,y) := \max_{\vartheta\in\mathrm{E}(K;\varphi)} p(\vartheta(x),\vartheta(y))$$
and note that the family $P' = \{p' \mid p \in P\}$ satisfies all the conditions of Definition 1.3 (iii). Therefore, the extension $q\colon (K;\varphi) \to (L;\mathrm{id}_L)$ is pseudoisometric. Moreover, each fiber semigroup $\mathrm{E}_l(K;\varphi)$ is a compact topological group.

3) Let $(K;\varphi)$ be a dynamical system and $q\colon (K;\varphi) \to (K;\varphi)$ the isometric extension given by the identity map. Then for each $k \in K$
$$\mathrm{E}_k(K;\varphi) = \begin{cases} \{\mathrm{id}_{\{k\}}\} & \text{if } k \text{ is recurrent,}\\ \emptyset & \text{otherwise.}\end{cases}$$

4) Let $(\mathbb{D};\varphi)$ be the rotation with varying velocity on the disc $\mathbb{D}$, i.e., $\mathbb{D} = \{z \in \mathbb{C} \mid |z| \leqslant 1\}$ and
$$\varphi\colon \mathbb{D} \to \mathbb{D}, \quad r\mathrm{e}^{2\pi\mathrm{i}\alpha} \mapsto r\mathrm{e}^{2\pi\mathrm{i}(\alpha+r)}.$$
Then
$$q\colon (\mathbb{D};\varphi) \to \left([0,1];\mathrm{id}_{[0,1]}\right), \quad r\mathrm{e}^{2\pi\mathrm{i}\alpha} \mapsto r$$
defines an isometric extension. For rational $r \in [0,1]$, the fiber semigroup $\mathrm{E}_r(K;\varphi)$ is a finite cyclic group and for irrational r it is isomorphic to $\mathbb{T} = \{z \in \mathbb{C} \mid |z| = 1\}$.

5) Take $\alpha \in \mathbb{T}$ and let $(\mathbb{T}^2;\varphi_\alpha)$ be the corresponding skew rotation, i.e.,
$$\varphi_\alpha\colon \mathbb{T}^2 \to \mathbb{T}^2, \quad (x,y) \mapsto (\alpha x, xy).$$
Then, denoting by $\psi_\alpha\colon \mathbb{T} \to \mathbb{T}$ the rotation $\psi_\alpha(x) = \alpha x$, consider the isometric extension
$$q\colon (\mathbb{T}^2;\varphi_\alpha) \to (\mathbb{T};\psi_\alpha), \quad (x,y) \mapsto x.$$
If α is irrational, each fiber semigroup is ismorphic to $\mathbb{T}$, but if α is rational their structure is more complex.

6) Consider the system $(L;\psi)$ given by $L := [0,1]$ and $\psi(x) := x^2$ and define a group extension $q\colon (K;\varphi) \to (L;\psi)$ by setting $K := [0,1]\times\mathbb{Z}^2$, $\varphi(x,g) := (\psi(x), g+1)$, and $q(x,g) := x$ for $(x,g) \in K$. Then q is an isometric extension of the nonminimal system $(L;\psi)$ for which $\mathrm{E}_l(K;\varphi) = \emptyset$ for every $l \in (0,1)$.

None of the above systems is minimal, yet the extensions still fit into the picture of Knapp's result. This suggests that minimality might be inessential for his result and given the restrictiveness of minimality, it is clear that the question of the necessity of minimality is important.

A (semi)groupoid approach. Among the best-behaved examples in Examples 1.6 is example 2): It can be shown (see [Ede19]) that q is necessarily open and that the fiber groups form an open, compact group bundle. Moreover, under a topological assumption on the map $q\colon K \to L$, this group bundle completely determines the system $(K;\varphi)$ or, more precisely, the extension $q\colon (K;\varphi) \to (L;\mathrm{id}_L)$. However, the rotation on the unit disc in 4) shows that one cannot expect to obtain a compact group bundle in general and example 6) demonstrates that the approach of fiber semigroups fails severely if the systems are not pointwise recurrent.

The underlying reason is that whereas most of the information about a single system $(K;S,\varphi)$ is contained in the semigroup $\{\varphi_s \mid s \in S\}$ and, by extension, the Ellis semigroup $\mathrm{E}(K;S,\varphi)$, most of the information about an extension $q\colon (K;S,\varphi) \to (L;S,\psi)$ is encoded in the set

$$\mathcal{S}(q) := \{\varphi_s|_{K_l} \mid s \in S, l \in L\}$$

of all restrictions to fibers. This is reflected by the fact that most notions for extensions can be defined using only the set $\mathcal{S}(q)$, as one readily verifies for Definition 1.3. In light of this and the apparent limitations of the enveloping semigroup $\mathrm{E}(K;S)$ and the derived fiber semigroups $\mathrm{E}_l(K;S)$, a more promising approach appears to be the construction of an enveloping structure for $\mathcal{S}(q)$. However, $\mathcal{S}(q)$ is no longer a semigroup since only some elements can be composed, making $\mathcal{S}(q)$ a *semigroupoid* and the object we are looking for an *enveloping semigroupoid*.

Following [MMMM13, Definitions 2.1, 2.2, and 2.17], we recall the definition of groupoids and semigroupoids.

Definition 1.7. A *semigroupoid* consists of a set $\mathcal{S}$, a set $\mathcal{S}^{(2)} \subseteq \mathcal{S}\times\mathcal{S}$ of composable pairs, and a product map $\cdot\colon \mathcal{S}^{(2)} \to \mathcal{S}$ that is associative in the sense that

(i) if $(g_1,g_2),(g_2,g_3) \in \mathcal{S}^{(2)}$, then $(g_1\cdot g_2, g_3), (g_1, g_2\cdot g_3) \in \mathcal{S}^{(2)}$ and $(g_1\cdot g_2)\cdot g_3 = g_1\cdot(g_2\cdot g_3)$.

We usually abbreviate $g\cdot h$ by gh if there is no room for confusion. We call a semigroupoid $\mathcal{G}$ a *groupoid* if there is an *inverse map* ${}^{-1}\colon \mathcal{G}\to\mathcal{G}$ such that, additionally, for each $g \in \mathcal{G}$

(ii) $(g^{-1},g) \in \mathcal{G}^{(2)}$ and if $(g,h) \in \mathcal{G}^{(2)}$, then $g^{-1}(gh) = h$,

(iii) $(g,g^{-1}) \in \mathcal{G}^{(2)}$ and if $(h,g) \in \mathcal{G}^{(2)}$, then $(hg)g^{-1} = h$.

If $\mathcal{G}$ is a groupoid,

$$\mathcal{G}^{(0)} := \{g^{-1}g \mid g \in \mathcal{G}\}$$

is called the *unit space* of $\mathcal{G}$ and the maps

$$s\colon \mathcal{G}\to\mathcal{G}^{(0)},\quad g\mapsto g^{-1}g,$$
$$r\colon \mathcal{G}\to\mathcal{G}^{(0)},\quad g\mapsto gg^{-1}$$

are called the *source* and *range maps* of $\mathcal{G}$. For $u,v \in \mathcal{G}^{(0)}$, we write $\mathcal{G}_u := s^{-1}(u)$, $\mathcal{G}^v := r^{-1}(v)$, and $\mathcal{G}_u^v := \mathcal{G}_u \cap \mathcal{G}^v$. A groupoid is *transitive* if $\mathcal{G}_u^v \neq \emptyset$ for all $u,v \in \mathcal{G}^{(0)}$ and *a group bundle* if $\mathcal{G}_u^v = \emptyset$ for all $u, v \in \mathcal{G}^{(0)}$ with $u \neq v$. If $\mathcal{G}$ is a group bundle, we write $p := r = s$.

A *topological semigroupoid* is a semigroupoid $(\mathcal{S}, \mathcal{S}^{(2)}, \cdot)$ with a topology on $\mathcal{S}$ such that the product map is continuous. We define *topological groupoids* analogously by demanding that the inverse map be continuous, too. Finally, *subsemigroupoids* and *subgroupoids* of a given semigroupoid or groupoid are defined in a straightforward way.

Example 1.8. Let $q\colon (K;S) \to (L;S)$ be an extension of topological dynamical systems. Then $\mathcal{S}(q)$ is a semigroupoid with the set of composable pairs

$$\mathcal{S}^{(2)}(q) := \{(\vartheta, \eta) \in \mathcal{S}(q) \times \mathcal{S}(q) \mid \operatorname{im}(\eta) \subseteq \operatorname{dom}(\vartheta)\}$$

and the product map given by composition. If $S = G$ is a group, this yields a groupoid with the canonical inverse map.

Example 1.9. Let $q\colon K \to L$ be a continuous surjection of compact spaces and consider the set

$$\mathcal{T} := \mathrm{C}_q^q(K, K) := \bigcup_{l, l' \in L} \mathrm{C}\left(K_l, K_{l'}\right).$$

Then setting

$$\mathcal{T}^{(2)}(q) := \{(\vartheta, \eta) \in \mathcal{T} \times \mathcal{T} \mid \operatorname{im}(\eta) \subseteq \operatorname{dom}(\vartheta)\}$$

and again taking composition as the product map, we obtain the *semigroupoid of continuous fiber maps* of the extension. In the situation of Example 1.8, $\mathrm{C}_q^q(K, K)$ contains $\mathcal{S}(q)$ as a subsemigroupoid.

Example 1.10. Given a groupoid $\mathcal{G}$, the subgroupoid

$$\operatorname{Iso}(\mathcal{G}) := \{g \in \mathcal{G} \mid s(g) = r(g)\}$$

of $\mathcal{G}$ becomes a group bundle called the *isotropy bundle* of $\mathcal{G}$.

Example 1.11. Let L be a set. Then $\mathcal{G}_L := L \times L$ is a groupoid with the set of composable pairs

$$\mathcal{G}_L^{(2)} := \{((x, y), (y, z)) \,|\, x, y, z \in L\},$$

a product map defined by $(x, y) \cdot (y, z) := (x, z)$, and the inverse map $(x, y) \mapsto (y, x)$. The groupoid $\mathcal{G}_L$ is called the *pair groupoid* of L and has the property that the equivalence relations on L can be identified with full subgroupoids of $\mathcal{G}_L$, where a subgroupoid is called *full* if it has the same unit space as its ambient groupoid.

In order to define an enveloping semigroupoid for $\mathcal{S}(q)$, it is necessary to form its closure in a larger semigroupoid with respect to a suitable topology. As the examples given above illustrate, a pointwise closure is ill-suited for studying equicontinuous extensions of nonminimal systems. However, we observe the following: For equicontinuous systems, it can be shown that the semigroup $\mathrm{E}(K; S, \varphi)$ coincides with the *uniform enveloping semigroup* $\mathrm{E}_{\mathrm{u}}(K; S, \varphi)$ defined as the closure of the semigroup $\{\varphi_s \mid s \in S\}$ with respect to the compact-open topology. Moreover, the topology of pointwise convergence and the compact-open topology then coincide on $\mathrm{E}(K; S, \varphi)$. Motivated by this observation, we introduce a relativized version of the compact-open topology on the space of continuous functions between fibers of two bundles in order to define the uniform enveloping semigroupoid $\mathcal{E}_{\mathrm{u}}(\mathcal{S}(q))$.

Definition 1.12. If $p\colon X \to L$ and $q\colon Y \to L'$ are continuous surjections onto compact spaces, set

$$\mathrm{C}_p^q(X,Y)_l^{l'} := \mathrm{C}\,(X_l, Y_{l'}) \quad \text{for } (l,l') \in L \times L'$$

and define the set of *continuous fiber maps* between $p\colon X \to L$ and $q\colon Y \to L'$ as

$$\mathrm{C}_p^q(X,Y) := \bigcup_{l \in L, l' \in L'} \mathrm{C}_p^q(K,X)_l^{l'}.$$

We define "source" and "range" maps

$$s\colon \mathrm{C}_p^q(X,Y) \to L, \quad r\colon \mathrm{C}_p^q(X,Y) \to L'$$

by setting

$$s(\vartheta) := l \quad \text{and} \quad r(\vartheta) := l' \quad \text{for } \vartheta \in \mathrm{C}_p^q(K,X)_l^{l'}.$$

If Y is a topological space and $q\colon Y \to \mathrm{pt}$ is the unique map onto a one-point space pt, we abbreviate $\mathrm{C}_p(X,Y) := \mathrm{C}_p^q(X,Y)$. Moreover, we write $\mathrm{C}_p(X) := \mathrm{C}_p(X,\mathbb{C})$.

In order to endow $\mathrm{C}_p^q(X,Y)$ with a topology, observe that an element $\vartheta \in \mathrm{C}_p^q(X,Y)$ may be identified with its graph $\mathrm{Gr}(\vartheta) \subseteq X \times Y$. Therefore, $\mathrm{C}_p^q(X,Y)$ may be regarded as a subspace of the space $\mathscr{C}(X \times Y)$ of closed subsets of $X \times Y$, on which there exist many topologies such as the Vietoris topology.

Definition 1.13. Let X be a topological space and $\mathscr{C}(X)$ the set of its nonempty closed subsets. The *Vietoris topology* on $\mathscr{C}(X)$ is the topology generated by the sets

$$U^- := \{A \in \mathscr{C}(X) \mid A \cap U \neq \emptyset\} \text{ and}$$
$$U^+ := \{A \in \mathscr{C}(X) \mid A \subseteq U\}$$

for open subsets $U \subseteq X$.

Remark 1.14. It is known that if X is a Hausdorff space, then so is $\mathscr{C}(X)$, see [Mic51, Theorem 4.9]. If X is compact, then $\mathscr{C}(X)$ is also compact, see [Mic51, Theorem 4.9] or [EE14, Proposition 5.A.3]. If, additionally, X is a metric space, the Vietoris topology coincides with the topology induced by the Hausdorff metric, see [Mic51, Theorem 3.4 and Proposition 3.6] or [EE14, Exercise 5.4].

Definition 1.15. If $p\colon X \to L$ and $q\colon Y \to L'$ are continuous surjections of topological spaces X and Y onto compact spaces L and L', we define the *relativized compact-open topology* on $\mathrm{C}_p^q(X,Y)$ to be the initial topology with respect to the map

$$\mathrm{Gr}\colon \mathrm{C}_p^q(X,Y) \to \mathscr{C}(X \times Y), \quad \vartheta \mapsto \mathrm{Gr}(\vartheta)$$

where $\mathscr{C}(X \times Y)$ is equipped with the Vietoris topology.

Next, we characterize the convergence of nets with respect to the relativized compact-open topology. For technical reasons, we limit ourselves to open bundles.

Proposition 1.16. *Let $p\colon K \to L$, $q\colon Y \to L'$ be continuous surjections onto compact spaces. Suppose that K is compact and p is open. For a net $(\vartheta_\alpha)_{\alpha\in A}$ in $\mathrm{C}_p^q(K,Y)$ and a $\vartheta \in \mathrm{C}_p^q(K,Y)$ the following assertions are equivalent.*

(a) $\lim_\alpha \vartheta_\alpha = \vartheta$ *with respect to the relativized compact-open topology.*

(b) *The following two conditions are satisfied.*

- $\lim_\alpha s(\vartheta_\alpha) = s(\vartheta)$.
- *If $(\varphi_\beta)_{\beta\in B}$ is a subnet of $(\varphi_\alpha)_{\alpha\in A}$, then*

$$\lim_\beta \vartheta_\beta(x_\beta) = \vartheta(x)$$

for every net $(x_\beta)_{\beta\in B}$ in K that converges to some $x \in K$ and satisfies $q(x_\beta) = s(\vartheta_\beta)$ for every $\beta \in B$.

In particular, the relativized compact-open topology is the coarsest topology on $\mathrm{C}_p^q(K,Y)$ such that the maps

$$s\colon \mathrm{C}_p^q(K,Y) \to L, \qquad \vartheta \mapsto s(\vartheta)$$
$$\mathrm{ev}\colon \mathrm{C}_p^q(K,Y) \times_{s,p} K \to Y, \quad (\vartheta, x) \mapsto \vartheta(x)$$

are continuous.

Remark 1.17. Note that a continuous surjection $q\colon K \to L$ between compact spaces K and L is open if and only if the following condition is fullfilled: For every convergent net $(l_\alpha)_{\alpha\in A}$ in L with limit $l \in L$ and every $x \in K_l$, there are a subnet $(l_\beta)_{\beta\in B}$ of $(l_\alpha)_{\alpha\in A}$ and a net $(x_\beta)_{\beta\in B}$ in K that converges to x and covers $(l_\beta)_{\beta\in B}$ in the sense that $q(x_\beta) = l_\beta$ for every $\beta \in B$. We will make use of this observation in the proof of Proposition 1.16 and at several occasions in the article.

Proof of Proposition 1.16. Suppose $(\vartheta_\alpha)_{\alpha\in A}$ converges to ϑ with respect to the relativized compact-open topology. Then for every $U \in \mathcal{U}(s(\vartheta))$,

$$\mathrm{Gr}(\vartheta) \subseteq p^{-1}(U) \times Y$$

and so it follows that $(s(\vartheta_\alpha))_{\alpha\in A}$ eventually lies in U. Since U was arbitrary, $s(\vartheta_\alpha) \to s(\vartheta)$. Now pick $x \in K_{s(\vartheta)}$ and a net $(x_\beta)_{\beta\in B}$ as in (b). Let $V \in \mathcal{U}(\vartheta(x))$ be an open neighborhood and set $U_0 := \vartheta^{-1}(V) \subseteq K_{s(\vartheta)}$. Then there is an open neighborhood $U \in \mathcal{U}(x)$ such that $U_0 = U \cap K_{s(\vartheta)}$. If $C \in \mathcal{U}(x)$ is a closed neighborhood of x with $C \subseteq U$, then

$$\mathrm{Gr}(\vartheta) \subseteq U \times V \cup C^c \times Y.$$

Since $x_\beta \to x$ and $\vartheta_\beta \to \vartheta$ with respect to the Vietoris topology, $(x_\beta)_{\beta\in B}$ eventually lies in C and $(\mathrm{Gr}(\vartheta_\beta))_{\beta\in B}$ eventually lies in $U\times V\cup C^c\times Y$. Thus, $(x_\beta, \vartheta_\beta(x_\beta))_{\beta\in B}$ eventually lies in $C \times V \subseteq U \times V$. Since $V \in \mathcal{U}(\vartheta(x))$ was arbitrary, it follows that $\vartheta_\beta(x_\beta) \to \vartheta(x)$.

Conversely, suppose (b) holds and assume that $(\vartheta_\alpha)_{\alpha\in A}$ does not converge to ϑ with respect to the relativized compact-open topology. Passing to a subnet, we may assume that there is an open subset $U \subseteq K \times Y$ such that

- $\vartheta \subseteq U$ and $\vartheta_\alpha \nsubseteq U$ for every $\alpha \in A$ or

- $\vartheta \cap U \neq \emptyset$ and $\vartheta_\alpha \cap U = \emptyset$ for every $\alpha \in A$.

In the first case, we find $x_\alpha \in K_{s(\vartheta_\alpha)}$ such that $(x_\alpha, \vartheta_\alpha(x_\alpha)) \notin U$ for each $\alpha \in A$. Again passing to a subnet, we may assume that $(x_\alpha)_{\alpha\in A}$ converges to some $x \in K$. By (b) we then obtain that $x \in K_{s(\vartheta)}$ and $\lim_\alpha \vartheta_\alpha(x_\alpha) = \vartheta(x)$. But since $(x, \vartheta(x)) \in U$ we find an $\alpha \in A$ with $(x_\alpha, \vartheta_\alpha(x_\alpha)) \in U$, a contradiction.

In the second case, we pick $x \in K_{s(\vartheta)}$ with $(x, \vartheta(x)) \in U$. Since q is open and $\lim_\alpha s(\vartheta_\alpha) = s(\vartheta)$, we may assume—by passing to a subnet—that there is a net $(x_\alpha)_{\alpha\in A}$ in K converging to x such that $x_\alpha \in K_{s(\vartheta_\alpha)}$ for each $\alpha \in A$. But then $(x, \vartheta(x)) = \lim_\alpha(x_\alpha, \vartheta_\alpha(x_\alpha))$ and therefore there is an $\alpha_0 \in A$ such that $(x_\alpha, \vartheta_\alpha(x_\alpha)) \cap U \neq \emptyset$, another contradiction. Hence, $\mathrm{Gr}(\vartheta_\alpha) \to \mathrm{Gr}(\vartheta)$ with respect to the Vietoris topology. □

Remark 1.18. It follows from Proposition 1.16 that if L consists only of a single point, then the relativized compact-open topology on $\mathrm{C}_q(K, Y) = \mathrm{C}(K, Y)$ coincides with the compact-open topology on $\mathrm{C}(K, Y)$.

After these preparations, we now regard the semigroupoid $\mathrm{C}_q^q(K, K)$ of fiber maps introduced in Example 1.9 as a topological semigroupoid with respect to the relativized compact-open topology. This allows to define the uniform enveloping semigroupoid of a set of fiber maps.

Definition 1.19. Let $q\colon K \to L$ be an open, continuous surjection of compact spaces and $\mathcal{F}$ be a subset of the topological semigroupoid $\mathrm{C}_q^q(K, K)$. Then the *uniform enveloping semigroupoid* $\mathcal{E}_\mathrm{u}(\mathcal{F})$ of $\mathcal{F}$ is defined to be the smallest closed subsemigroupoid of $\mathrm{C}_q^q(K, K)$ containing $\mathcal{F}$. If $q\colon (K; S) \to (L; S)$ is an extension of topological dynamical systems, we call

$$\mathcal{E}_\mathrm{u}(q) := \mathcal{E}_\mathrm{u}(\mathcal{S}(q))$$

the *uniform enveloping semigroupoid* of the extension.

Remark 1.20. Note that this definition makes sense since the intersection of a family of closed subsemigroupoids of a topological semigroupoid is again a closed subsemigroupoid. Also, the definition of the uniform enveloping semigroupoid of an extension $q\colon (K; S) \to (L; S)$ is more intricate than that of the Ellis semigroup $\mathrm{E}(K; S)$: The Ellis semigroup is defined as the closure of a semigroup and it turns out that this closure is automatically again a semigroup. The following example and Example 1.27 demonstrate that this is not true for $\mathcal{E}_\mathrm{u}(q)$ because it takes into account the global orbit structure of a system.

Example 1.21. Consider the invertible dynamical systems $(L; \psi)$ defined by $L := [-1, 1]$, $\psi(x) := \mathrm{sign}(x)x^2$ and $(K; \varphi)$ given by $K := [-1, 1] \times \mathbb{Z}_2$, $\varphi(x, g) := (\psi(x), g + 1)$. Then

$$q\colon (K; \varphi) \to (L; \psi), \quad (x, g) \mapsto x$$

is isometric. The uniform enveloping semigroupoid of q is given by

$$\mathcal{E}_\mathrm{u}(q) = \left\{\vartheta_{x,y,h} \,\middle|\, x, y \in L, h \in \mathbb{Z}_2\right\}$$

where $\vartheta_{x,y,h}$ denotes the function

$$\vartheta_{x,y,h}\colon K_x \to K_y, \quad (x,g) \mapsto (y, g+h).$$

In contrast to this,

$$\begin{aligned}\overline{\mathcal{S}(q)} = \mathcal{S}(q) &\cup \{\vartheta_{x,0,h}, \vartheta_{0,x,h} \mid x \in [-1,1], h \in \mathbb{Z}_2\} \cup \left\{\vartheta_{-1,y,h}, \vartheta_{y,-1,h} \,\middle|\, y \in [-1,0], h \in \mathbb{Z}_2\right\} \\ &\cup \left\{\vartheta_{1,y,h}, \vartheta_{y,1,h} \,\middle|\, y \in [0,1], h \in \mathbb{Z}_2\right\}.\end{aligned}$$

Thus, the inclusion $\overline{\mathcal{S}(q)} \subseteq \mathcal{E}(q)$ is generally strict.

As pointed out above, one of the key facts required for the proof of Theorem 1.2 is that for invertible equicontinuous systems the uniform enveloping semigroup is in fact a compact topological group. In the setting of extensions this raises the question: When is the uniform enveloping semigroupoid actually a compact groupoid? As a first step to address this problem, we observe that the groupoid property follows automatically for invertible systems, once we have ensured compactness.

Proposition 1.22. *Assume that $q\colon (K;G) \to (L;G)$ is an open extension of topological dynamical systems. If $\mathcal{E}_\mathrm{u}(q)$ is compact, then it is a compact groupoid, i.e., every $\vartheta \in \mathcal{E}_\mathrm{u}(q)$ has an inverse $\vartheta^{-1} \in \mathcal{E}_\mathrm{u}(q)$ and the mapping ${}^{-1}\colon \mathcal{E}_\mathrm{u}(q) \to \mathcal{E}_\mathrm{u}(q)$ is a homeomorphism.*

Proof. Consider the set M of all elements $\vartheta \in \mathcal{E}_\mathrm{u}(q)$ having an inverse ϑ^{-1} in $\mathcal{E}_\mathrm{u}(q)$. Then M is certainly closed under compositions and contains $\mathcal{S}(q)$. To see that $M = \mathcal{E}_\mathrm{u}(q)$ it therefore suffices to show that M is closed in $\mathcal{E}_\mathrm{u}(q)$. Pick a net $(\vartheta_\alpha)_{\alpha\in A}$ in M converging to $\vartheta \in \mathcal{E}_\mathrm{u}(q)$. Passing to a subnet we may assume that $(\vartheta_\alpha^{-1})_{\alpha\in A}$ converges to some element $\varrho \in \mathcal{E}_\mathrm{u}(q)$. Using Proposition 1.16 and the openness of q we conclude that $\varrho = \vartheta^{-1}$. This shows $M = \mathcal{E}_\mathrm{u}(q)$. Moreoverm if $(\vartheta_\alpha)_{\alpha\in A}$ is a net in $\mathcal{E}_\mathrm{u}(q)$ converging to some $\vartheta \in \mathcal{E}_\mathrm{u}(q)$, then a similar argument shows that ϑ^{-1} is the only cluster point of the net $(\vartheta_\alpha^{-1})_{\alpha\in A}$. □

Remark 1.23. If $q\colon (K;G) \to (L;G)$ is an open extension and $\mathcal{E}_\mathrm{u}(q)$ is a groupoid, then its unit space is

$$\mathcal{E}_\mathrm{u}(q)^{(0)} = \{\mathrm{id}_{K_l} \mid l \in L\}.$$

In the following we identify $\mathcal{E}_\mathrm{u}(q)^{(0)}$ with L. The source and range maps s and r then coincide with the restrictions of the mappings $r, s\colon \mathrm{C}_q^q(K,K) \to L$ defined in Definition 1.12 to the subspace $\mathcal{E}_\mathrm{u}(q)$.

We now try to characterize the compactness of the uniform enveloping semigroupoid by investigating when a set is (pre)compact in the relativized compact-open topology. Recall that if K is a compact space and Y is a uniform space, the precompactness of a subset $\mathcal{F} \subseteq \mathrm{C}(K,Y)$ is characterized by the classical Arzelà-Ascoli theorem: $\mathcal{F}$ is precompact if and only if $\mathcal{F}$ is equicontinuous and $\mathrm{im}(\mathcal{F}) = \bigcup_{f\in\mathcal{F}} \mathrm{im}(f)$ is precompact in Y. In what follows, we relativize the notion of equicontinuity and prove a generalization of the Arzelà-Ascoli theorem to compact bundles.

Definition 1.24. Let $p\colon K \to L$, $q\colon Y \to L'$ be continuous surjections onto compact spaces, K be compact, and Y be a uniform space. A subset $\mathcal{F} \subseteq \mathrm{C}_p^q(K,Y)$ is called *relatively (uniformly) equicontinuous* if for each $U \in \mathcal{U}_Y$ there is a $V \in \mathcal{U}_K$ such that $(\vartheta(x), \vartheta(y)) \in U$ for every $\vartheta \in \mathcal{F}$ and every $(x, y) \in V \cap K \times_L K$.

Theorem 1.25. *Let $p\colon K \to L$, $q\colon Y \to L'$ be continuous surjections onto compact spaces, K be compact, and Y be a Hausdorff uniform space. If p is open, then a subset $\mathcal{F} \subseteq \mathrm{C}_p^q(K, Y)$ is precompact if and only if the following two conditions are fulfilled.*

(i) $\mathrm{im}(\mathcal{F}) \subseteq Y$ *is precompact.*

(ii) $\mathcal{F}$ *is relatively equicontinuous.*

Proof. Suppose that (i) and (ii) hold. In view of Remark 1.14, it suffices to show that the closure $\overline{\mathrm{Gr}(\mathcal{F})}$ in $\mathscr{C}(K \times Y)$ is in fact contained in $\mathrm{Gr}(\mathrm{C}_p^q(K, Y))$. So we pick $C \in \overline{\mathrm{Gr}(\mathcal{F})}$ and show that $C = \mathrm{Gr}(\vartheta)$ for some $\vartheta \in \mathrm{C}_p^q(K, Y)$.

Let $(\vartheta_\alpha)_{\alpha \in A}$ be a net in $\mathcal{F}$ such that $\mathrm{Gr}(\vartheta_\alpha) \to C$ with respect to the Vietoris toplogy. First, let $(x, y) \in C$ and set $l := p(x)$, $l' := q(x)$. We claim that $C \subseteq K_l \times Y_{l'}$: If $U \in \mathcal{U}_L(l)$ and $V \in \mathcal{U}_{L'}(l')$ are open neighborhoods of l and l', then

$$C \cap p^{-1}(U) \times q^{-1}(V) \neq \emptyset.$$

Thus, there is an $\alpha_0 \in A$ such that for all $\alpha \geqslant \alpha_0$

$$\mathrm{Gr}(\vartheta_\alpha) \cap p^{-1}(U) \times q^{-1}(V) \neq \emptyset.$$

Since $\vartheta_\alpha \in \mathrm{C}_p^q(K, Y)$, it follows that $\mathrm{Gr}(\vartheta_\alpha) \subseteq p^{-1}(U) \times q^{-1}(V)$ for $\alpha \geqslant \alpha_0$ and hence that $C \subseteq \overline{p^{-1}(U)} \times \overline{q^{-1}(V)}$. Since U and V were arbitrary, $C \subseteq K_l \times Y_{l'}$.

Since p is open, it follows that for every $x \in K_l$ there is a $y \in Y_{l'}$ such that $(x, y) \in C$: Use Remark 1.17 and the compactness of $\mathrm{im}(\mathcal{F})$ to find a subnet $(\mathrm{Gr}(\vartheta_\beta))_{\beta \in B}$ and a net $(x_\beta)_{\beta \in B}$ such that $(x_\beta)_{\beta \in B}$ converges to x, $p(x_\beta) = s(\vartheta_\beta)$ for every $\beta \in B$, and $(\vartheta_\beta(x_\beta))_{\beta \in B}$ converges to some $y \in Y$. Since $(\mathrm{Gr}(\vartheta_\beta))_{\beta \in A}$ converges to C with respect to the Vietoris topology, this then shows that $(x, y) \in C$. In order to see that C is, in fact, the graph of a function $\vartheta\colon K_l \to Y_{l'}$, assume that $(x, y), (x, y') \in C$. Then there are nets $(x_\alpha, \vartheta_\alpha(x_\alpha))_{\alpha \in A}$, $(x'_\alpha, \vartheta_\alpha(x'_\alpha))_{\alpha \in A}$ converging to (x, y) and (x, y'). It then follows from the equicontinuity of $\mathcal{F}$ that the nets $(\vartheta_\alpha(x_\alpha))_{\alpha \in A}$ and $(\vartheta_\alpha(x'_\alpha))_{\alpha \in A}$ have the same limits. This shows that $y = y'$, i.e., there is a function $\vartheta\colon K_l \to Y_{l'}$ with $C = \mathrm{Gr}(\vartheta)$. Since K_l is compact and $Y_{l'}$ is Hausdorff, the closed graph theorem shows that ϑ is continuous, i.e., $\vartheta \in \mathrm{C}_p^q(K, Y)$. Hence, $\mathcal{F}$ is compact.

For the converse implication, we may assume $\mathcal{F}$ to be compact. Using Proposition 1.16, it is then easy to see that $\mathrm{im}(\mathcal{F})$ is compact. If $\mathcal{F}$ were not relatively equicontinuous, we would find a net $((\vartheta_\alpha, x_\alpha, x'_\alpha))_{\alpha \in A}$ in $\mathcal{F} \times_L K \times_L K$ and a $U \in \mathcal{U}_Y$ such that $\lim_\alpha x_\alpha = \lim_\alpha x'_\alpha$ and $(\vartheta_\alpha(x_\alpha), \vartheta_\alpha(x'_\alpha)) \notin U$ for every $\alpha \in A$ which clearly contradicts the compactness of $\mathcal{F}$. Thus, $\mathcal{F}$ is equicontinuous. □

This characterization of compactness in turn allows to characterize equicontinuous extensions and also derive an operator theoretic characterization of such extensions via the Koopman operator.

Corollary 1.26. *For an open extension $q\colon (K;S) \to (L;S)$ of topological dynamical systems the following assertions are equivalent.*

(a) *q is equicontinuous.*

(b) *$\overline{\mathcal{S}(q)} \subseteq \mathrm{C}_q^q(K,K)$ is compact.*

(c) *$\{T_s f \mid s \in S\} \subseteq \mathrm{C}_q(K)$ is relatively equicontinuous for every $f \in \mathrm{C}(K)$.*

(d) *$\overline{\{T_s f \mid s \in S\}} \subseteq \mathrm{C}_q(K)$ is compact for every $f \in \mathrm{C}(K)$.*

In particular, if $\mathcal{E}_{\mathrm{u}}(q)$ is compact, q is necessarily equicontinuous. The following example shows that the converse is generally not true because of the generally strict inclusion $\overline{\mathcal{S}(q)} \subseteq \mathcal{E}_{\mathrm{u}}(q)$ noted in Remark 1.20 and Example 1.21.

Example 1.27. Let $L_0 := [0,\infty)$ and

$$\psi_0\colon L_0 \to L_0, \quad \psi_0(x) := \lfloor x \rfloor + (x - \lfloor x \rfloor)^2$$

as well as $K_0 := L_0 \times \mathbb{Z}_2$ and

$$\varphi_0\colon K_0 \to K_0, \quad \varphi_0(x,g) = (\psi_0(x), g+1).$$

Then $q_0\colon K_0 \to L_0$, $(x,h) \mapsto x$ is continuous and intertwines φ_0 and ψ_0. Since ψ_0, φ_0, and q are proper, they extend canonically to the one-point compactifications $K := K_0 \cup \{\infty_{K_0}\}$ and $L := L_0 \cup \{\infty_{L_0}\}$ of K_0 and L_0 and thereby yield an extension $q\colon (K;\varphi) \to (L;\psi)$ of invertible topological dynamical systems. It is easy to see that $\overline{\mathcal{S}(q)}$ is compact since

$$\overline{\mathcal{S}(q)} \subseteq \{\vartheta_\infty\} \cup \bigcup_{n\in\mathbb{N}_0} \left\{\vartheta_{x,y,g} \,\middle|\, x,y \in [n,n+1], g \in \mathbb{Z}_2\right\}$$

where for $x,y \in L$ and $g \in \mathbb{Z}_2$, we define $\vartheta_{x,y,g}$ and ϑ_x as

$$\begin{aligned} \vartheta_{x,y,g}\colon K_x &\to K_y, & (x,h) &\mapsto (y, g+h),\\ \vartheta_x\colon K_x &\to \{\infty_{K_0}\}, & (x,h) &\mapsto \infty_{K_0}. \end{aligned}$$

However,

$$\mathcal{E}_{\mathrm{u}}(q) = \left\{\vartheta_{x,y,g} \,\middle|\, x,y \in L_0, g \in \mathbb{Z}_2\right\} \cup \{\vartheta_x \mid x \in L\}$$

and since ϑ_x is not invertible for $x \neq \infty_{L_0}$, $\mathcal{E}_{\mathrm{u}}(q)$ is neither compact nor a groupoid.

Thus, in order to characterize the compactness of $\mathcal{E}_{\mathrm{u}}(q)$, a more restrictive property than equicontinuity is needed. In the remainder of this section, we show that, under appropriate assumptions, $\mathcal{E}_{\mathrm{u}}(q)$ is compact if and only if the extension q is pseudoisometric. We start with the easy implication.

Proposition 1.28. *Let $q\colon (K;G) \to (L;G)$ be an open pseudoisometric extension. Then $\mathcal{E}_{\mathrm{u}}(q)$ is a compact groupoid.*

Proof. Pick a set P as in Definition 1.3 (iii) and consider the set

$$\mathrm{I}(P) := \left\{\vartheta \in \mathrm{C}_q^q(K,K) \,\middle|\, \begin{array}{l} \vartheta\colon K_{s(\vartheta)} \to K_{r(\vartheta)} \text{ is bijective and for all } p \in P,\\ x,y \in K_{s(\vartheta)} \text{ one has } p(\vartheta(x),\vartheta(y)) = p(x,y) \end{array}\right\}.$$

By Theorem 1.25, $\mathrm{I}(P)$ is a compact (semi)groupoid containing $\mathcal{S}(q)$ and therefore $\mathcal{E}_\mathrm{u}(q) \subseteq \mathrm{I}(P)$ is itself a compact semigroupoid. It follows from Proposition 1.22 above that it is in fact a groupoid. □

In order to prove a partial converse to Proposition 1.28, we need to take a closer look at Examples 1.6 3): In the case of the isometric extension $\mathrm{id}_K\colon (K;G) \to (K;G)$, $\mathcal{E}_\mathrm{u}(\mathrm{id}_K)$ is a compact groupoid by Proposition 1.28. However, $\mathrm{C}_{\mathrm{id}_K}^{\mathrm{id}_K}(K,K)$ can be identified with the pair groupoid $K \times K$ and as seen in Example 1.11, the closed subgroupoid $\mathcal{E}_\mathrm{u}(\mathrm{id}_K) \subseteq K \times K$ is then just a closed equivalence relation on K. As we will show, this equivalence relation yields the maximal trivial factor of the system.

Definition 1.29. Let $(K;S)$ be a topological dynamical system. Then a factor $(L;S)$ of $(K;S)$ is called *trivial* if S acts trivially on L. A trivial factor $(M;S)$ is called a *maximal trivial factor* of $(K;S)$ if any trivial factor of $(K;S)$ also is a factor of $(M;S)$.

Any system $(K;S)$ has a maximal trivial factor which is unique up to isomorphy, cf. [Aus88, Exercise 9.2]. We therefore speak of *the* maximal trivial factor (K_fix, S) of a system. Up to isomorphy, the trivial factors of $(K;S)$ can be identified with closed, S-invariant equivalence relations of K and the maximal trivial factor then corresponds to the smallest closed, invariant equivalence relation. This equivalence relation can also be characterized in terms of the *fixed space*

$$\mathrm{fix}(T_\varphi) := \left\{ f \in \mathrm{C}(K) \,\middle|\, \forall\, s \in S\colon T_{\varphi_s} f = f \right\}$$

of the Koopman representation of S on $\mathrm{C}(K)$: The equivalence relation

$$\sim_\mathrm{fix} := \left\{ (x,y) \in K \times K \,\middle|\, \forall\, f \in \mathrm{fix}(T_\varphi)\colon f(x) = f(y) \right\}$$

certainly is closed and invariant and it can be shown that it coincides with the smallest such equivalence relation.

Lemma 1.30. *Let $(K;G)$ be a topological dynamical system. Then $K/\mathcal{E}_\mathrm{u}(\mathrm{id}_K)$ is the maximal trivial factor K_fix of $(K;G)$.*

Proof. By construction, $\mathcal{E}_\mathrm{u}(\mathrm{id}_K)$ is the smallest closed equivalence relation of $K \times K$ that contains the orbit equivalence relation

$$\mathcal{S}(\mathrm{id}_K) = \{(x,y) \in K \times K \mid \exists\, g \in G\colon y = gx\}.$$

Thus, $K/\mathcal{E}_\mathrm{u}(\mathrm{id}_K)$ is the maximal trivial factor of $(K;G)$. □

Of special interest to us is the case when the maximal trivial factor is a point, i.e., if every invariant function $f \in \mathrm{C}(K)$ is constant. As proposed by M. Haase, we call such systems topologically ergodic.

Definition 1.31. A topological dynamical system $(K;S,\varphi)$ is called *topologically ergodic* if its maximal trivial factor is a point, i.e., if the fixed space $\mathrm{fix}(T_\varphi)$ consists only of constant functions.

The following observation links topological ergodicity with transitivity of the uniform enveloping groupoid. Recall that a groupoid $\mathcal{G}$ is *transitive* if $\mathcal{G}_u^v \neq \emptyset$ for all $u, v \in \mathcal{G}^{(0)}$.

Proposition 1.32. *Let $q\colon (K;G) \to (L;G)$ be an open extension such that $\mathcal{E}_{\mathrm{u}}(q)$ is a compact groupoid. Then $\mathcal{E}_{\mathrm{u}}(q)$ is transitive if and only if $(L;G)$ is topologically ergodic.*

Proof. The set

$$\mathcal{S} := \{\vartheta \in \mathcal{E}_{\mathrm{u}}(q) \mid (s(\vartheta), r(\vartheta)) \in \mathcal{E}_{\mathrm{u}}(\mathrm{id}_L)\}$$

is a closed subsemigroupoid of $\mathcal{E}_{\mathrm{u}}(q)$ containing $\mathcal{S}(q)$ and therefore $\mathcal{S} = \mathcal{E}_{\mathrm{u}}(q)$. The mapping

$$(s, r)\colon \mathcal{E}_{\mathrm{u}}(q) \to \mathcal{E}_{\mathrm{u}}(\mathrm{id}_L), \quad \vartheta \mapsto (s(\vartheta), r(\vartheta)).$$

is continuous and its image is a compact subsemigroupoid of $\mathcal{E}_{\mathrm{u}}(\mathrm{id}_L)$ containing $\mathcal{S}(\mathrm{id}_L)$, wich means that (s, r) is surjective. Transitivity of $\mathcal{E}_{\mathrm{u}}(q)$ is just a reformulation of the equivalence relation $\mathcal{E}_{\mathrm{u}}(\mathrm{id}_L)$ being all of $L \times L$. □

We are now ready to state the final result of this section.

Theorem 1.33. *Let $q\colon (K;G) \to (L;G)$ be an open extension of dynamical systems such that $(L;G)$ is topologically ergodic. Then the following assertions are equivalent.*

(a) *q is pseudoisometric.*

(b) *$\mathcal{E}_{\mathrm{u}}(q)$ is a compact groupoid.*

Proof. The implication (a) $\implies$ (b) was already established more generally in Proposition 1.28, so assume that $\mathcal{E}_{\mathrm{u}}(q)$ is a compact groupoid. We mimick the construction of Examples 1.6 2). That is, let P be a family of pseudometrics generating the topology of K. Then for $p \in P$, define

$$p'\colon K \times_L K \to [0, \infty), \quad p'(x, y) := \max_{\substack{\vartheta \in \mathcal{E}_{\mathrm{u}}(q) \\ s(\vartheta) = q(x)}} p(\vartheta(x), \vartheta(y))$$

Then the family $P' := \{p' \mid p \in P\}$ generates the topology of K_l for each $l \in L$ since $\mathcal{E}_{\mathrm{u}}(q)_l$ is compact. Moreover, since the range and source map of a compact transitive groupoid are open (see Proposition 2.3 below), each p' is continuous and one readily verifies the invariance of the p'. □

Remark 1.34. Let $q\colon (K;G) \to (L;G)$ be an open extension of topological dynamical systems such that $(L;G)$ is topologically ergodic. If K is metrizable, then the proof above reveals that q is isometric if and only if it is pseudoisometric.

2. Haar systems and relatively invariant measures

As discussed in the introduction, it is known that equicontinuous extensions of minimal topological dynamical systems always admit a relatively invariant measure. We recall the definition (cf. [Gla75, Section 3]). Here and henceforth, we write δ_l for the Dirac measure of a point l in a compact space L.

Definition 2.1. Let $q\colon (K;S) \to (L;S)$ be an extension of topological dynamical systems. Moreover let $(\mathrm{P}(K);S)$ be the induced dynamical system on the space $\mathrm{P}(K)$ of probability measures equipped with the weak* topology. A *relatively invariant measure* for q is a morphism

$$\mu\colon (L;S) \to (\mathrm{P}(K);S), \quad l \mapsto \mu_l$$

such that $q_*\mu_l = \delta_l$ for all $l \in L$. We call μ *fully supported* if $\operatorname{supp}\mu_l = K_l$ for every $l \in L$.

Using the (uniform) Ellis semigroup, it can be shown that any equicontinuous, minimal system $(K;G)$ has a unique invariant probability measure which is the pushforward of the Haar measure on the compact group $\mathrm{E}(K;G)$. We use the uniform enveloping semigroupoid in order to prove a generalization of this in terms of relatively invariant measures for pseudoisometric extensions. To that end, we require Haar systems, a generalization of Haar measures to group bundles and more generally groupoids, see [Ren80, Definition 2.2].

Definition 2.2. Let $\mathcal{G}$ be a compact group bundle and for $u \in \mathcal{G}^{(0)}$ let m_u be the Haar measure on the fiber group $\mathcal{G}_u$. Then $\mathcal{G}$ *has a continuous Haar system* if the mapping

$$\mathcal{G}^{(0)} \to \mathbb{C}, \quad u \mapsto \int f \,\mathrm{d}m_u$$

is continuous for each $f \in \mathrm{C}(\mathcal{G})$.

It is known, that a compact group bundle $\mathcal{G}$ has a continuous Haar system if and only if the mapping $p\colon \mathcal{G} \to \mathcal{G}^{(0)}$ is open (see [Ren91, Lemma 1.3]). The following result shows that this always holds for isotropy bundles of compact transitive groupoids (see Definition 1.7 above for the definition).

Proposition 2.3. *Let $\mathcal{G}$ be a compact transitive groupoid. Then (s,r), s, and r are open and so is the restriction p of s and r to* $\mathrm{Iso}(\mathcal{G})$. *In particular, the isotropy bundle* $\mathrm{Iso}(\mathcal{G})$ *has a continuous Haar system.*

Proof. We start with the restrictions to $\mathrm{Iso}(\mathcal{G})$: Pick $g \in \mathrm{Iso}(\mathcal{G})$ and set $u := p(g) \in \mathcal{G}^{(0)}$. Moreover, let $(u_\alpha)_{\alpha\in A}$ be a net in $\mathcal{G}^{(0)}$ converging to u. For each $\alpha \in A$ we there is an $h_\alpha \in \mathcal{G}_u^{u_\alpha}$ and by passing to a subnet, we may assume that $\lim_\alpha h_\alpha = h \in \mathcal{G}_u^u$. But then $g = \lim_\alpha h_\alpha(h^{-1}gh)h_\alpha^{-1}$ and so we have found a net $(g_\alpha)_{\alpha\in A}$ in $\mathrm{Iso}(\mathcal{G})$ that converges to g and satisfies $r(g_\alpha) = u_\alpha$ for every $\alpha \in A$. Thus, r is open.

To show that (s,r), s, and r are open, it suffices to show that (s,r) is open, so let $g \in \mathcal{G}$ and $(u_\alpha, v_\alpha)_{\alpha\in A}$ be a net in $\mathcal{G}^{(0)} \times \mathcal{G}^{(0)}$ converging to $(u,v) = (s(g), r(g))$. Since $\mathcal{G}$ is transitive, there is a net $(h_\alpha)_{\alpha\in A}$ in $\mathcal{G}$ with $s(h_\alpha) = u_\alpha$ and $r(h_\alpha) = v_\alpha$ for each $\alpha \in A$. By compactness of $\mathcal{G}$, we may assume that $(h_\alpha)_{\alpha\in A}$ converges to some element $h \in \mathcal{G}$ in $\mathcal{G}_{s(g)}^{r(g)}$. Set $\gamma := gh^{-1} \in \mathrm{Iso}(\mathcal{G})_{r(g)}$ and, using the openness result for the isotropy bundle, find—after possibly passing to a subnet—a net $(\gamma_\alpha)_{\alpha\in A}$ in $\mathrm{Iso}(\mathcal{G})$ with $p(\gamma_\alpha) = v_\alpha$ for each $\alpha \in A$. Then the net $(\gamma_\alpha h_\alpha)_{\alpha\in A}$ converges to g and satisfies $(s(h_\alpha), r(h_\alpha)) = (u_\alpha, v_\alpha)$ for each $\alpha \in A$. Hence, (s,r) is open. □

Corollary 2.4. *Let $q\colon (K;G) \to (L;G)$ be an open and pseudoisometric extension and $(L;G)$ be topologically ergodic. Then the isotropy bundle $\mathrm{Iso}(\mathcal{E}_\mathrm{u}(q))$ has a continuous Haar system.*

For an open pseudoisometric extension $q\colon (K;G) \to (L;G)$ we want to "push forward" the continuous Haar system of $\mathrm{Iso}(\mathcal{E}_\mathrm{u}(q))$ to K in order to obtain a relatively invariant measure. This works if the fiber groups $\mathrm{Iso}(\mathcal{E}_\mathrm{u}(q))$ act transitively on the fibers of q and the next result characterizes when this is the case.

Proposition 2.5. *For an open pseudoisometric extension $q\colon (K;G) \to (L;G)$ of topological dynamical systems the following assertions are equivalent.*

(a) *The group action of $\mathrm{Iso}(\mathrm{E}(q))_l$ on K_l is transitive for every $l \in L$.*
(b) $T_q \operatorname{fix}(T_\psi) = \operatorname{fix}(T_\varphi)$.

For the proof we need the following lemma.

Lemma 2.6. *Let $q\colon (K;G) \to (L;G)$ be an open pseudoisometric extension of topological dynamical systems. Then the following assertions hold.*

(i) *The map*
$$\mathcal{E}_\mathrm{u}(q) \times_{s,q} K \to \mathcal{E}_\mathrm{u}(\mathrm{id}_K), \quad (\vartheta, x) \mapsto (x, \vartheta(x))$$
is a continuous surjection.

(ii) *If $q_{\mathrm{fix}}^K\colon K \to K_{\mathrm{fix}}$ is the factor map to the maximal trivial factor K_{fix} of $(K;G)$, then*
$$\mathrm{Iso}(\mathcal{E}_\mathrm{u}(q))_{q(x)}(x) = q^{-1}(q(x)) \cap (q_{\mathrm{fix}}^K)^{-1}(q_{\mathrm{fix}}^K(x))$$
for every $x \in K$.

Proof. For (i), notice that the set
$$\mathcal{S} := \{\vartheta \in \mathcal{E}_\mathrm{u}(q) \mid \forall\, x \in K_{s(\vartheta)}\colon (x, \vartheta(x)) \in \mathcal{E}_\mathrm{u}(\mathrm{id}_K)\}$$
is a closed subsemigroupoid of $\mathcal{E}_\mathrm{u}(q)$ that contains $\mathcal{S}(q)$ and therefore $\mathcal{S} = \mathcal{E}_\mathrm{u}(q)$. Clearly, the mapping
$$\mathcal{E}_\mathrm{u}(q) \times_{s,q} K \to \mathcal{E}_\mathrm{u}(\mathrm{id}_K), \quad (\vartheta, x) \mapsto (x, \vartheta(x))$$
is continuous. Since its image is a compact subsemigroupoid of $\mathcal{E}_\mathrm{u}(\mathrm{id}_K)$ containing $\mathcal{S}(\mathrm{id}_K)$, (i) holds. Part (i) and Lemma 1.30 then yield (ii). □

Proof of Proposition 2.5. By Lemma 2.6 (ii), we obtain that $\mathrm{Iso}(\mathrm{E}(q))_l$ acts transitively on K_l for every $l \in L$ if and only if $q^{-1}(q(x)) \subseteq q_{\mathrm{fix}}^{-1}(q_{\mathrm{fix}}(x))$ for every $x \in K$. Now consider the following commutative diagram.

$$\begin{tikzcd} & K \arrow[dl, "q"'] \arrow[dr, "q_{\mathrm{fix}}^K"] & \\ L \arrow[dr, "q_{\mathrm{fix}}^L"'] & & K_{\mathrm{fix}} \arrow[dl, "p"] \\ & L_{\mathrm{fix}} & \end{tikzcd}$$

First, assume that $T_q \operatorname{fix}(T_\psi) = \operatorname{fix}(T_\varphi)$. Then p is injective and if $x \in K$, then

$$q^{-1}(q(x)) \subseteq q^{-1}((q_{\mathrm{fix}}^L)^{-1}(q_{\mathrm{fix}}^L(q(x)))) = (q_{\mathrm{fix}}^K)^{-1}(p^{-1}(p(q_{\mathrm{fix}}^K(x)))) = (q_{\mathrm{fix}}^K)^{-1}(q_{\mathrm{fix}}^K(x)).$$

Conversely, assume that $\mathrm{Iso}(\mathrm{E}(q))_l$ acts transitively on K_l for every $l \in L$ and pick $f \in \operatorname{fix}(T_\varphi)$. Then f takes a constant value c_l on K_l for every $l \in L$ and it is easy to see that $\widetilde{f}(l) := c_l$ for $l \in L$ defines a function $\widetilde{f} \in \operatorname{fix}(T_\psi) \subseteq \mathrm{C}(L)$ such that $T_q\widetilde{f} = f$. □

We now state the main result of this section.

Theorem 2.7. *Let $q\colon (K;G) \to (L;G)$ be an open pseudoisometric extension and $(K;G)$ be topologically ergodic. Then there is a unique relatively invariant measure for q. Moreover, this relatively invariant measure is fully supported.*

We first prove two lemmas.

Lemma 2.8. *Let $q\colon K \to L$ be a continuous open surjection between compact spaces and*

$$\mu\colon L \to \mathrm{P}(K), \quad l \mapsto \mu_l$$

a continuous map with $q_*\mu_l = \delta_l$ for every $l \in L$. Moreover, let $(f_\alpha)_{\alpha\in A}$ be a convergent net in $\mathrm{C}_q(K)$ with limit $f \in \mathrm{C}_q(K)$. Then

$$\lim_\alpha \int_{K_{s(f_\alpha)}} f_\alpha \, \mathrm{d}\mu_{s(f_\alpha)} = \int_{K_{s(f)}} f \, \mathrm{d}\mu_{s(f)}.$$

Proof. Choose $F \in \mathrm{C}(K)$ such that $F|_{K_{s(f)}} = f$. For each $\alpha \in A$ choose an $x_\alpha \in K_{s(f_\alpha)}$ such that

$$|f_\alpha(x_\alpha) - F(x_\alpha)| = \sup_{x\in K_{s(f_\alpha)}} |f_\alpha(x) - F(x)|.$$

For each subnet of $(f_\alpha)_{\alpha\in A}$ we then find a subnet $(f_\beta)_{\beta\in B}$ such that $x = \lim_\beta x_\beta$ exists in K. But then

$$\lim_\beta \sup_{x\in K_{s(f_\beta)}} |f_\beta(x) - F(x)| = \lim_\beta |f_\beta(x_\beta) - F(x_\beta)| = 0.$$

As a consequence,

$$\lim_\alpha \left| \int_{K_{s(f_\alpha)}} f_\alpha \, \mathrm{d}\mu_{s(f_\alpha)} - \int_{K_{s(f_\alpha)}} F \, \mathrm{d}\mu_{s(f_\alpha)} \right| \leqslant \lim_\alpha \sup_{x\in K_{s(f_\alpha)}} |f_\alpha(x) - F(x)| = 0,$$

which implies the claim. □

Lemma 2.9. *Let $q\colon (K;S) \to (L;S)$ be an open extension with relatively invariant measure μ. Then $\vartheta_*\mu_{s(\vartheta)} = \mu_{r(\vartheta)}$ for every $\vartheta \in \mathcal{E}_\mathrm{u}(q)$.*

Proof. The set

$$\mathcal{S} := \{\vartheta \in \mathcal{S} \mid \vartheta_*\mu_{s(\vartheta)} = \mu_{r(\vartheta)}\}$$

is a subsemigroupoid $\mathcal{E}_\mathrm{u}(q)$ containing $\mathcal{S}(q)$. We only have to check that it is closed. If $(\vartheta_\alpha)_{\alpha\in A}$ is a net in $\mathcal{S}$ converging to $\vartheta \in \mathcal{E}_\mathrm{u}(q)$ and $f \in \mathrm{C}(K)$, then $\lim_\alpha T_{\vartheta_\alpha} f = T_\vartheta f$ in $\mathrm{C}_q(K)$ and therefore $\lim_\alpha \langle T_{\vartheta_\alpha} f, \mu_{s(\vartheta_\alpha)}\rangle = \langle T_\vartheta f, \mu_{s(\vartheta)}\rangle$ by Lemma 2.8. Thus,

$$\langle f, \vartheta_*\mu_{s(\vartheta)}\rangle = \lim_\alpha \langle T_\vartheta f, \mu_{s(\vartheta)}\rangle = \lim_\alpha \langle f, (\vartheta_\alpha)_*\mu_{s(\vartheta_\alpha)}\rangle = \lim_\alpha \langle f, \mu_{r(\vartheta_\alpha)}\rangle = \langle f, \mu_{r(\vartheta)}\rangle.$$

This shows that $\vartheta \in \mathcal{S}$ and so $\mathcal{E}_\mathrm{u}(q) = \mathcal{S}$. □

Proof of Theorem 2.7. As usual, we denote the Haar measure on $\mathrm{Iso}(\mathcal{E}_\mathrm{u}(q))_l$ by m_l for $l \in L$. For $x \in K$, we denote by $\rho_x\colon\ \mathrm{Iso}(\mathcal{E}_\mathrm{u}(q))_{q(x)},\ g \mapsto gx$ the induced map onto the orbit of x. Now pick a point $x_l \in K_l$ for each $l \in L$ and set

$$\mu_l := (\rho_{x_l})_*(m_l).$$

It is clear from the transitivity of the group actions of $\mathrm{Iso}(\mathcal{E}_\mathrm{u}(q))_l$ on K_l that μ_l does not depend on the choice of $x_l \in K_l$ and that $\operatorname{supp}\mu_l = K_l$ for every $l \in L$. Moreover, $\vartheta_*\mu_{s(\vartheta)} = \mu_{r(\vartheta)}$ for every $\vartheta \in \mathcal{S}(q)$.

Now take $f \in \mathrm{C}(K)$. We show that $\lim_\alpha \mu_{l_\alpha}(f) = \mu_l(f)$ for every net $(l_\alpha)_{\alpha\in A}$ converging to some $l \in L$. By passing to a subnet, we may assume that there is a convergent net $(x_\alpha)_{\alpha\in A}$ in K with limit $x \in K$ that satisfies $q(x_\alpha) = l_\alpha$ for all $\alpha \in A$. Then $\rho_{x_\alpha} \to \rho_x$ with respect to the relativized compact-open topology and so $f\circ\rho_{x_\alpha} \to f\circ\rho_x$ with respect to the relativized compact-open topology. Therefore, Lemma 2.8 yields

$$\lim_{\alpha\in A}\langle f, \mu_{l_\alpha}\rangle = \lim_{\alpha\in A}\langle f\circ\rho_{x_\alpha}, m_{l_\alpha}\rangle = \langle f\circ\rho_x, m_l\rangle = \langle f, \mu_l\rangle\,.$$

Hence, $\mu\colon L \to \mathrm{P}(K)$ is continuous.

Finally, take any relatively invariant measure $\nu\colon L \to \mathrm{P}(K)$ for q and let $l \in L$. By Lemma 2.9 the measure ν_l is invariant with respect to the action of the fiber group $\mathrm{Iso}(\mathcal{E}_\mathrm{u}(q))_l$. Since a transitive action of a compact group is equicontinuous and minimal and therefore uniquely ergodic, $\nu_l = \mu_l$ and since $l \in L$ was arbitrary, μ is the unique relatively invariant measure for q. □

3. Representations of compact transitive groupoids

In this final section, we study the representation theory of compact transitive groupoids and apply it to the uniform enveloping (semi)groupoids of open pseudoisometric extensions. We start by recalling the concept of Banach bundles (see, e.g., [DG83, Definition 1.1] or [Gie82, Section 1 and Theorem 3.2]).

Definition 3.1. Let L be a compact space. A *Banach bundle* over L is a topological space E, called the *total space*, together with a continuous open surjection $p\colon E \to L$ with the following properties.

(i) Every fiber E_l is a Banach space.

(ii) The mappings

$$+\colon E \times_L E \to E, \quad (e, f) \mapsto e + f$$
$$\cdot\colon E \times E \to E, \quad (\lambda, e) \mapsto \lambda e$$

are continuous.

(iii) The norm mapping

$$\|\cdot\|\colon E \to [0, \infty), \quad e \mapsto \|e\|$$

is upper semicontinuous.

(iv) For each $l \in L$, the sets

$$\{e \in E \mid p(e) \in U, \|e\| < \varepsilon\}$$

for neighborhoods $U \subseteq L$ of l and $\varepsilon > 0$ define a neighborhood base of $0_l \in E_l$.

A Banach bundle E is

- *continuous* if the norm mapping $\|\cdot\|$ of (iii) is continuous,
- *of constant dimension n* for some $n \in \mathbb{N}_0$ if $\dim(E_l) = n$ for every $l \in L$.
- *of constant finite dimension* if it is of constant dimension n for some $n \in \mathbb{N}_0$.
- *locally trivial* if for each $l \in L$ there are a compact neighborhood W of l, $n \in \mathbb{N}_0$ and a homeomorphism $\Phi\colon p^{-1}(W) \to W \times \mathbb{C}^n$ with the following properties.
  - The diagram

$$\begin{tikzcd} p^{-1}(W) \arrow[rr, "\Phi"] \arrow[dr, "p"'] & & W \times \mathbb{C}^n \arrow[dl, "\mathrm{pr}_1"] \\ & W & \end{tikzcd}$$

    commutes where $\mathrm{pr}_1\colon W \times \mathbb{C}^n \to W$ is the prorjection onto the first component.
  - $\Phi|_{E_l}\colon E_l \to \{l\} \times \mathbb{C}^n$ is an isomorphism of vector spaces for every $l \in W$.
  - There are constants $c_1, c_2 > 0$ such that

$$c_1 \cdot \|e\| \leqslant \|\mathrm{pr}_2(\Phi(e))\| \leqslant c_2\|e\|$$

    for every $e \in p^{-1}(W)$ where $\mathrm{pr}_2\colon W \times \mathbb{C}^n \to \mathbb{C}^n$ is the projection onto the second component.

Moreover, we write

$$\Gamma(E) := \{\sigma \in \mathrm{C}(L, E) \mid p \circ \sigma = \mathrm{id}_L\}$$

for the *space of continuous sections of E*.

Remark 3.2. (i) If E is a Banach bundle over a compact space L, then $\Gamma(E)$ is canonically a module over $\mathrm{C}(L)$ and a Banach space with the norm defined by $\|\sigma\| := \sup_{l\in L} \|\sigma(l)\|_{E_l}$ for $\sigma \in \Gamma(E)$. Moreover, $\|f\sigma\| \leqslant \|f\| \cdot \|\sigma\|$ for all $f \in \mathrm{C}(L)$ and $\sigma \in \Gamma(E)$, i.e., $\Gamma(E)$ is a Banach module over $\mathrm{C}(L)$ (cf. [DG83, Chapter 2]).

(ii) If E is a continuous Banach bundle, then its total space is Hausdorff (see [Gie82, Proposition 16.4]).

(iii) A Banach bundle with finite-dimensional fibers which is locally trivial as a vector bundle (in the usual sense) is locally trivial as a Banach bundle (see [Gie82, Proposition 10.9]).

(iv) By [Gie82, Theorem 18.5], a Banach bundle of constant finite dimension has a Hausdorff total space if and only if it is locally trivial. In particular, every Banach bundle of constant finite dimension with a Hausdorff total space is locally trivial as a vector bundle and therefore its section space $\Gamma(E)$ is finitely generated and projective as a $\mathrm{C}(L)$-module by [Swa62, Theorem 2].

A *subbundle* of a Banach bundle E is a subset F of E together with the restricted mapping $p|_F\colon F \to L$ such that the following conditions are satisfied.

- $F_l = F \cap E_l$ is a closed linear subspace of E_l for every $l \in L$.
- The restricted mapping $p|_F$ is still open.

Under these conditions, F together with $p|_F$ in fact becomes a Banach bundle (see [Gie82, Section 8]).

There are plenty of examples of Banach bundles coming from differential geometry. Here we are interested in Banach bundles arising from surjections of compact spaces.

Example 3.3. Let $q\colon K \to L$ be an open continuous surjection between compact spaces. Then a moment's thought reveals that the relativized compact-open topology on $\mathrm{C}_q(K)$ agrees with the topology generated by the base

$$V(F, U, \varepsilon) := \{f \in \mathrm{C}_q(K) \mid s(f) \in U, \|f - F|_{K_{s(f)}}\| < \varepsilon\}$$

for $F \in \mathrm{C}(K)$, open $U \subseteq L$, and $\varepsilon > 0$ (considered, e.g., in [Kna67, p. 30]). Together with the canonical mapping $p\colon \mathrm{C}_q(K) \to L$, the space $\mathrm{C}_q(K)$ becomes a continuous Banach bundle over L. Moreover, the mapping

$$\mathrm{C}(K) \to \Gamma(E), \quad F \mapsto [l \mapsto F_l]$$

is an isometric isomorphism of Banach modules over $\mathrm{C}(L)$.

Next, we introduce the notion of continuous representations for topological groupoids (cf. Definition 3.1 of [Bos11]).

Definition 3.4. Let $\mathcal{G}$ be a topological groupoid. A *continuous representation* T of $\mathcal{G}$ on a Banach bundle E over $\mathcal{G}^{(0)}$ is a familiy of bounded invertible operators

$$T(g)\colon E_{s(g)} \to E_{r(g)}$$

for $g \in \mathcal{G}$ such that

- $T(gh) = T(g)T(h)$ for all $(g,h) \in \mathcal{G}^{(2)}$,
- $T(g^{-1}) = T(g)^{-1}$ for every $g \in \mathcal{G}$,

and

$$\mathcal{G} \times_{s,p} E \to E, \quad (g, v) \mapsto T(g)v$$

is continuous. A subset F of E is $\mathcal{G}$-*invariant* if $T(g)(F \cap E_{s(g)}) \subseteq F \cap E_{r(g)}$ for every $g \in \mathcal{G}$.

Proposition 3.5. *Let $q\colon (K;G) \to (L;G)$ be an open extension of topological dynamical systems. Then $T(\vartheta)f := f \circ \vartheta^{-1}$ for $f \in \mathrm{C}(K_{\mathrm{s}(\vartheta)})$ and $\vartheta \in \mathcal{E}_{\mathrm{u}}(q)$ defines a continuous representation of $\mathcal{E}_{\mathrm{u}}(q)$ on the continuous Banach bundle $\mathrm{C}_q(K)$.*

Proof. We only check that the mapping

$$\mathcal{E}_{\mathrm{u}}(q) \times_{s,p} \mathrm{C}_q(K) \to \mathrm{C}_q(K), \quad (\vartheta, f) \mapsto T(\vartheta)f$$

is continuous since the remaining assertions are obvious. Pick a net $((\vartheta_\alpha, f_\alpha))_{\alpha\in A}$ in $\mathcal{E}_{\mathrm{u}}(q) \times_{s,p} E$ converging to $(\vartheta, f) \in \mathcal{E}_{\mathrm{u}}(q) \times_{s,p} E$. We have to show that $T(\vartheta_\alpha)f_\alpha$ converges to $T(\vartheta)f$ with respect to the relativized compact-open topology.

Let $((\vartheta_\beta, f_\beta))_{\beta\in B}$ be a subnet and $(x_\beta)_{\beta\in B}$ be a convergent net in K with limit $x \in K$ that satisfies $p(x_\beta) = r(\vartheta_\beta)$ for every $\beta \in B$. Then $\lim_\beta \vartheta_\beta^{-1}(x_\beta) = \vartheta^{-1}(x)$. Since $\lim_\beta f_\beta = f$,

$$\lim_\beta f_\beta\left(\vartheta_\beta^{-1}\left(x_\beta\right)\right) = f\left(\vartheta^{-1}(x)\right).$$

This shows that T is continuous. □

We now state our main theorem: a version of Theorem 1.1 for compact transitive groupoids. Here, a subset F of a Banach bundle E over a compact space L is called *fiberwise dense* if $F \cap E_l$ is dense in E_l for every $l \in L$. The notion of a *fiberwise total set* is defined analogously.

Theorem 3.6. *Let T be a continuous representation of a compact transitive groupoid $\mathcal{G}$ on a Banach bundle E over the unit space $\mathcal{G}^{(0)}$. Then the union of all invariant subbundles of constant finite dimension is fiberwise dense in E. If, moreover, $\mathcal{G}$ is abelian, then the union of all invariant subbundles of constant dimension one is fiberwise total in E.*

Remark 3.7. Notice that if E has a Hausdorff total space (in particular, if E has continuous norm), then the subbundles in Theorem 3.6 are locally trivial (see Remark 3.2 (iv)).

Proof of Theorem 3.6. Note first that each invariant subbundle F such that $\dim F_u < \infty$ for some $u \in \mathcal{G}^{(0)}$ already has constant finite dimension since $T(g) \in \mathscr{L}(F_{s(g)}, F_{r(g)})$ is an isomorphism of Banach spaces for each $g \in \mathcal{G}_u$ and $\mathcal{G}$ is transitive by assumption.

Now fix $u \in \mathcal{G}^{(0)}$. The idea is to prove the claim by using the transitivity of the groupoid and Theorem 1.1 for the compact isotropy group $\mathcal{G}_u^u = \mathrm{Iso}(\mathcal{G})_u$. So let F_u be a $\mathcal{G}_u^u$-invariant, finite-dimensional subspace of E_u as provided by Theorem 1.1. For every $g \in \mathcal{G}_u$, we define a subspace $F(g) \subseteq E_{r(g)}$ by setting $F(g) := T(g)F_u$. Then $F(g)$ only depends on $r(g)$, for if $r(g) = r(g')$ and $s(g) = s(g') = u$, then

$$F(g') = T(g')F_u = T(g)T(g^{-1}g')F_u = T(g)F_u$$

since $g^{-1}g' \in \mathcal{G}_u^u$ and F_u is $\mathcal{G}_u^u$-invariant. Thus, we can define a prospective subbundle $F \subseteq E$ by setting $F_v := F(g)$ for any $v \in \mathcal{G}^{(0)}$ and $g \in \mathcal{G}_u^v$ and defining

$$F := \bigcup_{v \in \mathcal{G}^{(0)}} F_v.$$

This set is invariant since if $g \in \mathcal{G}^{(2)}$ and $h \in \mathcal{G}_u^{s(g)}$, then

$$T(g)F_{s(g)} = T(g)T(h)F_u = T(gh)F_u = F_{r(gh)} = F_{r(g)}.$$

To show that F is a subbundle of E, it suffices to check that $p|_F \colon F \to \mathcal{G}^{(0)}$ is open. So let $f \in F$ and $(v_\alpha)_{\alpha \in A}$ be a convergent net in $\mathcal{G}^{(0)}$ with limit $v := p(f) \in \mathcal{G}^{(0)}$. Since $(s, r) \colon \mathcal{G} \to \mathcal{G}^{(0)} \times \mathcal{G}^{(0)}$ is open by Proposition 2.3, we may assume—after passing to a subnet—that there is a net $(g_\alpha)_{\alpha \in A}$ with limit v such that $s(g_\alpha) = v$ and $r(g_\alpha) = v_\alpha$ for every $\alpha \in A$. The net $(T(g_\alpha)f)_{\alpha \in A}$ then is a net over $(v_\alpha)_{\alpha \in A}$ that converges to f, showing that $p|_F$ is open.

Since $l \in L$ was arbitrary, it follows from Theorem 1.1 applied to the fiber groups of the isotropy bundle $\mathrm{Iso}(\mathcal{G})$ of $\mathcal{G}$, that the union of all invariant subbundles of constant finite dimension is fiberwise dense in E. If $\mathcal{G}$ is abelian, the fibers of the bundles considered above are at most one-dimensional and using [EFHN15, Corollary 15.18] for the fiber groups yields the claim. □

We now apply Theorem 3.6 to the representations of uniform enveloping groupoids of pseudoisometric extensions and obtain a generalization of Knapp's result stated in the introduction. Recall that a module Γ over a commutative unital ring R is projective if there is an R-module $\widetilde{\Gamma}$ such that $\Gamma \oplus \widetilde{\Gamma}$ is a free R-module, i.e., has a basis.

Theorem 3.8. *Let $q \colon (K; \varphi) \to (L; G)$ be an open extension such that $(L; G)$ is topologically ergodic. Then the following assertions are equivalent.*

(a) *q is pseudoisometric.*

(b) *The union of all finitely generated and projective closed T_φ-invariant $\mathrm{C}(L)$-submodules of $\mathrm{C}(K)$ is dense in $\mathrm{C}(K)$.*

We first prove the following two lemmas.

Lemma 3.9. *Let L be a compact space and E a Banach bundle over L. If $\Gamma(E)$ is finitely generated and projective as a $\mathrm{C}(L)$-module, then E is locally trivial.*

Proof. By the Serre-Swan theorem (see [Swa62]) we find a locally trivial vector bundle F over L and a $\mathrm{C}(L)$-module isomorphism $T \colon \Gamma(F) \to \Gamma(E)$. Equip F with any mapping

$\|\cdot\|\colon F \to [0,\infty)$ turning F into a Banach bundle (these always exist, see [Swa62, Lemma 2]). Then by Remark 3.2 (iii) F is also locally trivial as a Banach bundle. If $l \in L$ and $\sigma \in \Gamma(F)$ with $\sigma(l) = 0$, then we find $h \in \mathrm{C}(L)$ with $h(l) = 0$ and $\tau \in \Gamma(F)$ such that $\sigma = h\tau$ by [Swa62, Corollary 3]. But then $T\sigma(l) = h(l)(T\tau)(l) = 0$. For every $l \in L$ we therefore obtain a well-defined linear map $\Phi_l\colon F_l \to E_l$ by setting $\Phi_l\sigma(l) := (T\sigma)(l)$ for $\sigma \in \Gamma(F)$ and $l \in L$. Moreover, since F_l is finite-dimensional, Φ_l is bounded for every $l \in L$. We show as in the proof of [Swa62, Theorem 1] that

$$\Phi\colon F \to E, \quad f \mapsto \Phi_{p(f)}f.$$

is continuous. Pick $l \in L$, a neighborhood $V \in \mathcal{U}_L(l)$ and sections $\sigma_1, \dots, \sigma_n \in \Gamma(F)$ such that $\sigma_1(\widetilde{l}), \dots, \sigma_n(\widetilde{l})$ define a base in $F_{\widetilde{l}}$ for every $\widetilde{l} \in V$. We then find continuous functions $h_1, \dots, h_n\colon p^{-1}(V) \to \mathbb{C}$ with $f = \sum_{j=1}^n h_j(f)\sigma_j(p(f))$ for every $f \in p^{-1}(V)$. But then $\Phi(f) = \sum_{j=1}^n h_j(f)(T\sigma_j)(p(f))$ for every $f \in p^{-1}(V)$. Since $T\sigma_j$ is continuous for every $j \in \{1, \dots, n\}$ we obtain that Φ is continuous. By [Gie82, Proposition 10.2] it is a morphism of Banach bundles and therefore T is a bounded operator. By the bounded inverse theorem we obtain that T is an isomorphism of Banach modules and therefore E and F are isomorphic as Banach bundles by [Gie82, Summary 10.18]. This shows that E is locally trivial. □

Lemma 3.10. *Let E be a Banach bundle over a compact space L which is locally trivial. If $M \subseteq E$ is a bounded subset, i.e., $\sup_{e\in M} \|e\|_{p(e)} < \infty$, then it is precompact.*

Proof. We may assume M to be closed. Now pick a net $(e_\alpha)_{\alpha\in A}$ in M. Passing to a subnet, we may assume that $(p(e_\alpha))_{\alpha\in A}$ converges to some $l \in L$. By choosing a local trivialization as in Definition 3.1, the claim reduces to the case of a trivial Banach bundle $L \times \mathbb{C}^n$ for which it is obvious. □

Proof of Theorem 3.8. If (a) holds, then applying Theorem 3.6 to the representation of $\mathcal{E}_\mathrm{u}(q)$ introduced in Proposition 3.5 yields that the $\mathcal{E}_\mathrm{u}(q)$-invariant subbundles of constant finite dimension are fiberwise dense in $\mathrm{C}_q(K)$. Since the Banach bundle $\mathrm{C}_q(K)$ is continuous, its total space is Hausdorff (see Remark 3.2 (ii)), and therefore these Banach bundles are locally trivial (see Remark 3.2 (iv)). Now take an invariant subbundle F of $\mathrm{C}_q(K)$ of constant finite dimension $n \in \mathbb{N}_0$. Then

$$\widetilde{\Gamma}(F) := \left\{s \in \Gamma(\mathrm{C}_q(K)) \,\middle|\, \forall\, l \in L\colon s(l) \in F_l\right\} \subseteq \Gamma(\mathrm{C}_q(K)),$$

is a $\mathrm{C}(L)$-submodule of $\Gamma(\mathrm{C}_q(K))$ which is isomorphic to $\Gamma(F)$ as a Banach module over $\mathrm{C}(L)$. In particular, $\widetilde{\Gamma}(F)$ is finitely generated and projective as a $\mathrm{C}(L)$-module (see Remark 3.2 (iv)) and closed in $\Gamma(\mathrm{C}_q(K))$.

Let M be the union of all modules $\widetilde{\Gamma}(F)$ where F is a $\mathcal{E}_\mathrm{u}(q)$-invariant subbundle of constant finite dimension. Then M is a $\mathrm{C}(L)$-submodule since the sum $F = F_1 + F_2$ of two invariant subbundles of F_1 and F_2 of constant finite dimension is again an invariant subbundle of constant finite dimension and $\widetilde{\Gamma}(F_1) + \widetilde{\Gamma}(F_2) \subseteq \widetilde{\Gamma}(F)$. Moreover, M is stalkwise dense in the sense of [Gie82, Definition 4.1] and via a Stone-Weierstraß theorem for bundles (see [Gie82, Corollary 4.3]), this implies that M is dense in $\Gamma(\mathrm{C}_q(K))$. Using the canonical isomorphy $\Gamma(\mathrm{C}_q(K)) \cong \mathrm{C}(K)$ we conclude that the union of all finitely generated and projective closed T_φ-invariant submodules is dense in $\mathrm{C}(K)$.

Now assume that (b) holds. By Theorem 1.25 it suffices to show that $\{f \circ \vartheta \mid \vartheta \in \mathcal{E}_\mathrm{u}(q)\}$ is a compact subset of $\mathrm{C}_q(K)$ for every f in a dense subset of $\mathrm{C}(K)$. We may assume that f is contained in a finitely generated and projective closed T_φ-invariant submodule $\widetilde{\Gamma}$ of $\mathrm{C}(K)$. For each $l \in L$, the subspace

$$F_l := \{h_l \mid l \in L\} \subseteq \mathrm{C}(K_l)$$

is finite-dimensional and, in particular, closed. By Example 3.3 and [Gie82, Theorem 8.6], this implies that

$$F := \bigcup_{l \in L} F_l$$

is a subbundle of the Banach bundle $\mathrm{C}_q(K)$ and that $\Gamma(F)$ is isomorphic to $\widetilde{\Gamma}$ as a Banach module over $\mathrm{C}(L)$. Therefore F is locally trivial by Lemma 3.9. We show that $F_{r(\vartheta)} \circ \vartheta \in F_{s(\vartheta)}$ for each $\vartheta \in \mathcal{E}_\mathrm{u}(q)$. For this, consider

$$\mathcal{S} := \left\{\vartheta \in \mathcal{E}_\mathrm{u}(q) \,\middle|\, F_{r(\vartheta)} \circ \vartheta \in F_{s(\vartheta)}\right\}.$$

It is clear, that $\mathcal{S}$ is a subsemigroupoid of $\mathrm{C}_q^q(K,K)$ containing $\mathcal{S}(q)$. We show that it is closed which implies $\mathcal{S} = \mathcal{E}_\mathrm{u}(q)$. Pick a net $(\vartheta_\alpha)_{\alpha\in A}$ in $\mathcal{S}$ converging to $\vartheta \in \mathcal{E}_\mathrm{u}(q)$ and $h \in \widetilde{\Gamma}$. Then $h_{r(\vartheta)} \circ \vartheta = \lim_\alpha h_{r(\vartheta_\alpha)} \circ \vartheta_\alpha$ in $\mathrm{C}_q(K)$. Since $(h_{r(\vartheta_\alpha)} \circ \vartheta_\alpha)_{\alpha\in A}$ is a bounded net in the locally trivial bundle F, we may assume—by passing to a subnet—that it converges to an element of F (see Lemma 3.10). Therefore $h_{r(\vartheta)} \circ \vartheta \in F$.

Using Lemma 3.10 once again, it now follows that $\{f \circ \vartheta \mid \vartheta \in \mathcal{E}_\mathrm{u}(q)\}$ is compact in $F \subseteq \mathrm{C}_q(K)$. □

Remark 3.11. Our main theorem Theorem 3.8 is applicable to many of the extensions in Examples 1.6. However, for some of the examples (e.g., the rotation on the disc over its maximal trivial factor in Examples 1.6 4)) the system $(L;G)$ is not topologically ergodic. A more general characterization of pseudoisometric extensions inluding these examples is the goal of future work.

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